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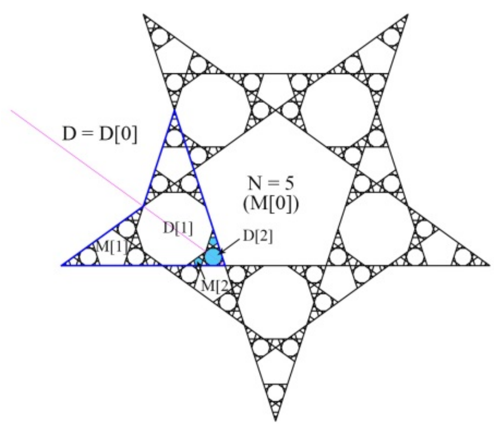
Summary of dynamics of the regular pentagon: N = 5

N = 5 and N = 8 are the only non-trivial regular cases where the singularity sets have been studied in detail. In [T] (1995) S. Tabachnikov derived the fractal dimension of W for N = 5 using 'normalization' methods and symbolic dynamics and in [S2] (2006) R. Schwartz used similar methods for N = 8. In [BC] (2011) Bedaride and Cassaigne reproduced Tabachnikov's results in the context of 'language' analysis and showed that N = 5 and N= 10 had equivalent sequences. In [H3] we gave an independent analysis of the temporal scaling of N = 5 based on difference equations and that will be reproduced here.

By the Twice-odd Lemma, the star polygon web for N = 5 shown here would be unchanged if the origin was moved from the cN to cD. The key issue in the geometry of W seems to be the interplay between the M[k] pentagons and the D[k] decagons . At each generation, these tiles are scaled by GenScale[5] = $\sqrt{5}-2$ from the previous generation. The region outlined in blue is invariant under τ^{10}, so it can be used as a 'template' for the growth of tiles. This region can be 'tiled' with scaled (and possibly rotated) copies of itself and each copy is anchored by a D[k]. The error in that tiling diminishes with each generation so our goal is to describe the 'temporal' growth of the D[k] tiles. To do this we will use difference equations to make explicit the relationship between the number of D[k] and M[k] tiles. These equations are shown in the table below. Click to enlarge the blue region.



Generation	decagons - d_n	pentagons - p_n
1	1	2
2	7 = $3d_1 + 2p_1$	10 = $6d_1 + 2p_1$
3	41 = $3d_2 + 2p_2$	62 = $6d_2 + 2p_2$
n	$d_n = 3d_{n-1} + 2p_{n-1}$	$p_n = 6d_{n-1} + 2p_{n-1}$

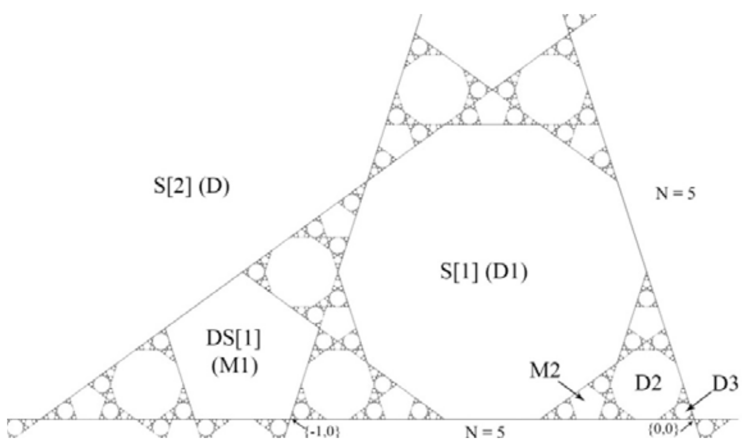
These two equations can be combined together to give a second-order equation: $d_n = 5d_{n-1} + 6d_{n-2}$. On the full 'star' region these same equations hold but the initial conditions are $d_1 = 5$ and $p_1 = 10$, so the solution is: $d_n = \frac{5}{7}[8\cdot 6^{n-1} + (-1)^n]$. This gives decagon center periods of 5, 35, 205, 1235,.. and pentagon periods of 10,50,310,… and shows that the ratio of the periods for the D's (and M's) approach 6 as in the 4k+1 conjecture. At each iteration, the error in tiling W with 6^k D[k]'s is reduced, so the 'box-counting' similarity dimension is Ln[6]/Ln[1/GenScale[5]] ≈ 1.24114. For compact, self-similar sets such as W this dimension is identical to the Hausdorff dimension. By the Twice-odd Lemma the singularity set for N = 10 will have the same fractal dimension. In Example 5.5 we will compare the singularity sets for the three quadratic polygons N = 5, 8 and 12.

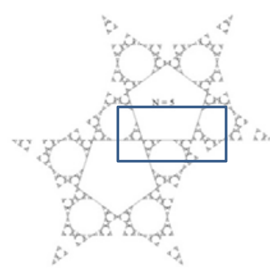
This is the first ‘quadratic’ polygon and also the first 4k+1 polygon. The only non-trivial scale is scale[2] = GenScale[5] = Tan[π/5]Tan[π/10] =$\sqrt{5}$ – 2. And The scaling fields S_5 and S_{10} are generated by x = GenScale[5] = Tan[π/5]Tan[π/10] =$\sqrt{5}$ - 2

hD/hN	hD1/hD	hD2/hD	hM1/hN	hM[2]/hN
x + 2	x	x^2	x	x^2

In keeping with the 4k+1 prediction, there appears to be well defined sequence of M[k] (and D[k|) tiles converging to GenStar[5] at the foot of D. Since this convergence takes place on the edges of D, it also occurs with N = 10 – so we will study that convergence later. The Twice-odd Lemma states that N = 5 and N = 10 will share the ‘same’ webs – but not necessarily the same dynamics. Since the fractal dimension of W depends on both geometric and temporal scaling we have to take into account the differences of dynamics – which is a relatively simple issue here.

By convention the N = 5 edge convergence will be studied at star[1] of N as shown here. This is primarily a D[k] convergence but of course the M[k]’s must converge to the origin at star[1] as well.





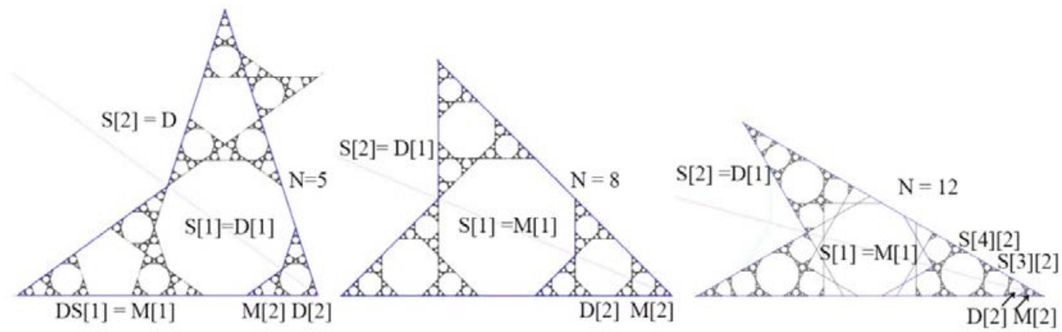
By the First Family Theorem the geometric scaling of these tiles is x = GenScale, so hM[k]/hN = x^k and hD[k]/hD = x^k. To find the fractal dimension it is also necessary to know the τ-period of these tiles. In general this is not a simple issue but the 4k+1 conjecture of [H5] predicts that the ratio of periods will be N + 1 and this can be verified using simple difference equations for pentagons and decagons in the ‘dart’ above between N and D.

Generation	decagons - d_n	pentagons - p_n
1	1 (D1)	2 (M[1] & matching one above it)
2	7 $= 3d_1 + 2p_1$	$10 = 6d_1 + 2p_1$
3	$41 = 3d_2 + 2p_2$	$62 = 6d_2 + 2p_2$
n	$d_n = 3d_{n-1} + 2p_{n-1}$	$p_n = 6d_{n-1} + 2p_{n-1}$

These same difference equations apply to the whole web with initial conditions 5 and 10 so the global solution is $d_n = \frac{5}{7}[8\cdot 6^{n-1} + (-1)^n]$ (See [H2]) This gives decagon center periods of 5, 35, 205, 1235,.. and pentagon periods of 10,50,310,…These difference equations show that the ratio of the periods for the D's (and M's) approach 6 as in the 4k+1 conjecture. These D's can be used to 'cover' the star region at all scales, so the Hausdorff-Besicovitch fractal dimension of W is Ln[6]/Ln[1/GenScale[5]] ≈ 1.24114.

This same temporal result is obtained in [H5] by the following argument.

N = 5, 8, 10 and 12 have $\varphi(N)/2 = 2$, so they have quadratic complexity- where the only non-trivial scale is GenScale[N]. Since the webs are naturally recursive, a single scale should yield a self-similar web and, under this assumption, we will derive the similarity dimension of these webs below – and hence the Hausdorff fractal dimension.



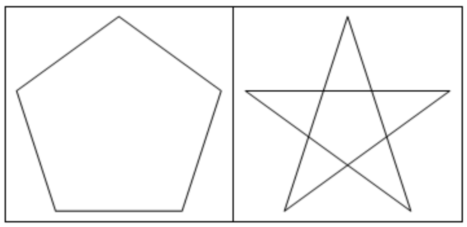
Since the geometric scaling is known, the only issue is the 'temporal' scaling – which describes the limiting growth in the number of tiles. In Example 5.1 we used the 'two dart' region on the left above to show that the D[k] for N = 5 have a temporal growth factor of 6. Here we will extend this analysis to the 'three-dart' structures of N = 8 and N = 12.

For self-similar webs this temporal scaling can be derived from a 'renormalization' process – where a representative portion of the web is scaled by GenScale[N] and mapped to itself under τ^k as shown by the magenta lines. As expected, the cases of N = 8 and 12 cases are closely related since their cyclotomic fields are generated by $\{\sqrt{2}, i\}$ and $\{\sqrt{3}, i\}$.

For N = 5 above we used difference equations to obtain the growth factor, but a simple geometric argument relating the M[k] and D[k] will suffice. The blue invariant region consists of two overlapping triangles or 'darts' and each dart is anchored by an M[k], so the M[k] scale by 2 with generations, and each M[k] is surrounded by 3 D[k+1]'s so the D[k] scale by 6 with generations. This matches the N + 1 temporal scaling predicted by difference equations, symbolic dynamics and the 4k+1 conjecture.

Projections

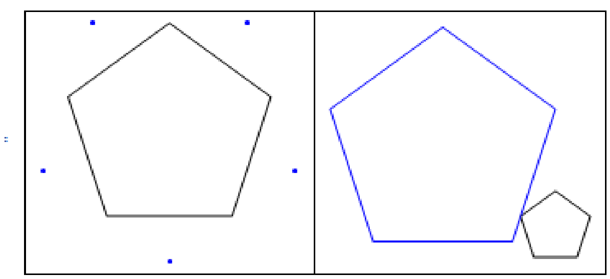
N = 5 has just one non-trivial projection, the P2 projection, which corresponds to remapping the vertices mod 2 as {1,3,5,2,4}, as shown on the right below. In the examples to follow we will show both the P1and P2 projections. The P1 projection will show the orbit of τ^2 - the 'return' map.



Example1: q1= cDad[1]= **cS[1]** ≈{-1.1755705045849462583,-0.38196601125010515179541}
This point is period 5 and since it is odd ,the projections will have the same period.

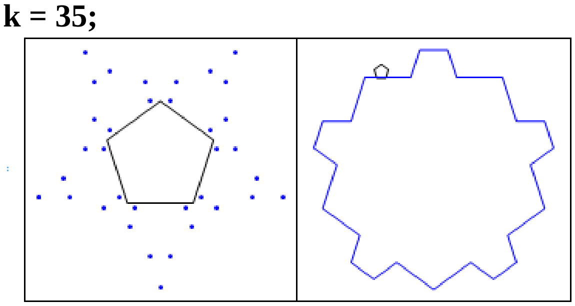
Ind = IND[q, 10]; k = 5;

GraphicsGrid[{{ Graphics[{poly[Mom], Blue, Point[PIM[q1,k,1]]}], Graphics[{poly[Mom], Blue,Line[PIM[q1,k,2]]}] }}, Frame->All]



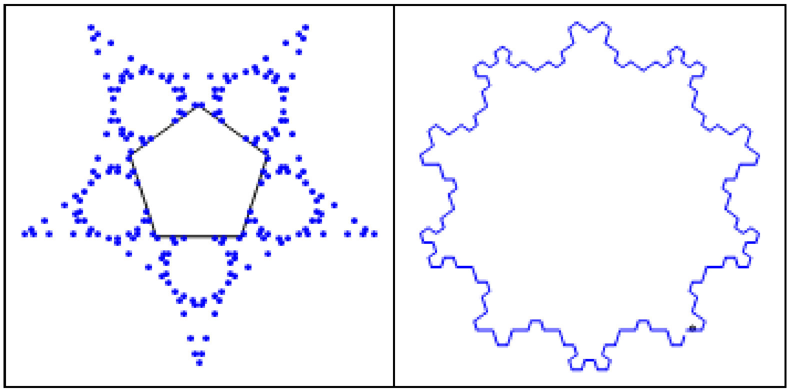
Example 2:q1= cDad[2] = **CFR[(2+GenScale)*GenScale^2]**
≈ {-2.1796275840160826576864,-0.70820393249936908922752}. Period 35

k = 35;



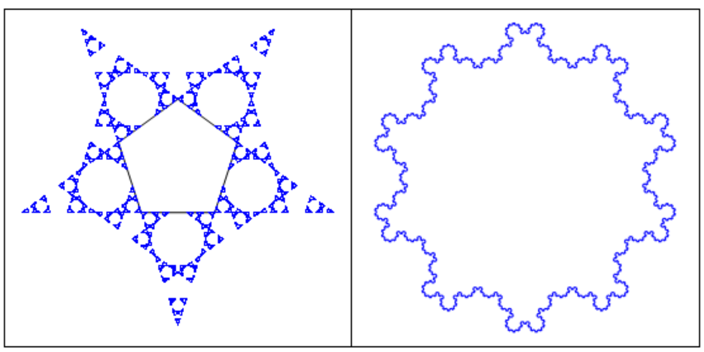
Example 3: q1= cDad[3] = **CFR[(2+GenScale)*GenScale^3]** ≈
{-2.4166533080517367211981493051,-0.78521825875241849129450280959}. Period 205

k = 205;

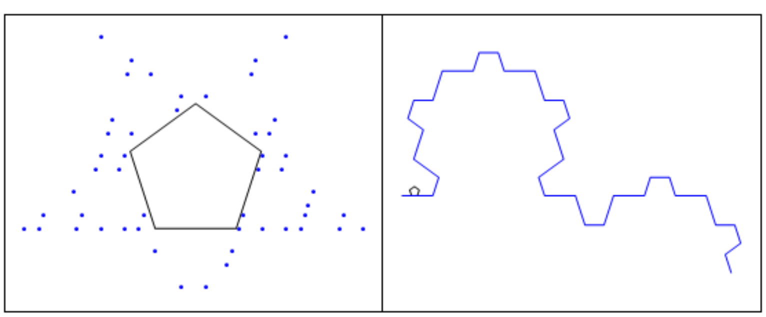


Example 4: q1= cDad[6] = CFR[(2+GenScale)*GenScale^6] ≈ {-2.48893470198405656489079346244,-0.80870390731219889339208617703O}

k = 44435;



k =50;

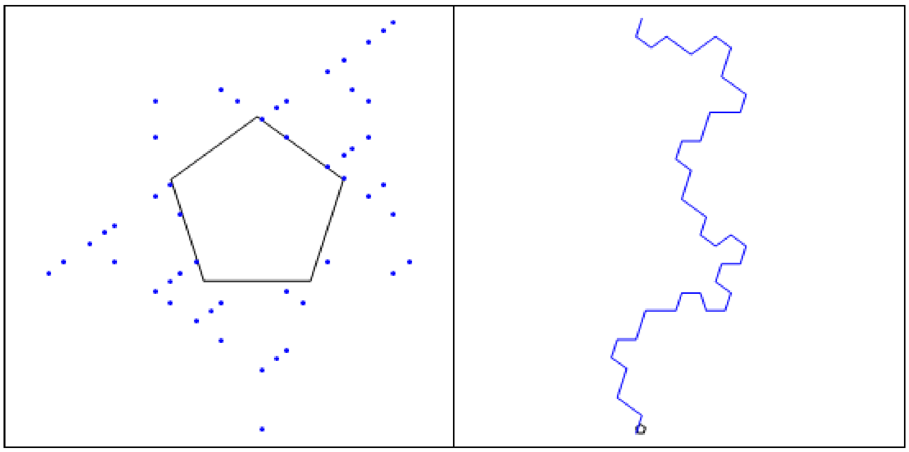


Example 6: The limiting orbit

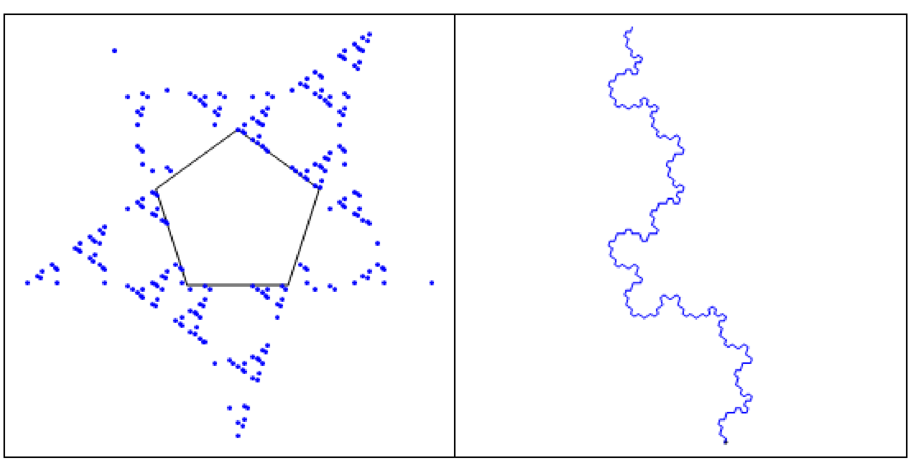
The limiting orbit of these Dads (or Moms) is a non-periodic orbit as explained above. In the limit, the P2 projection is a 'large-scale' fractal which can be converted to a 'normal' fractal by a rescaling on each generation. One point on this orbit is q**1= {Mom[[5]][[1]], Mom[[4]][[2]]}** ≈ {-0.95105651629515357211643933337 9,-0.80901699437494742410229341718 3}

Ind = IND[q, 1000000]; (*about 1 minute*)

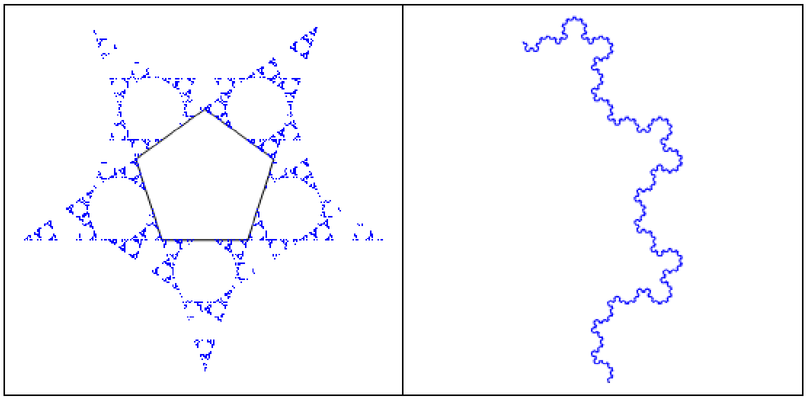
k = 50;



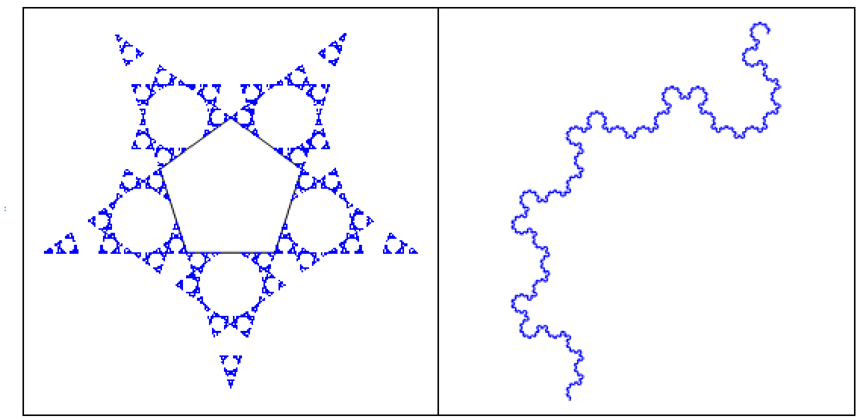
k = 250;



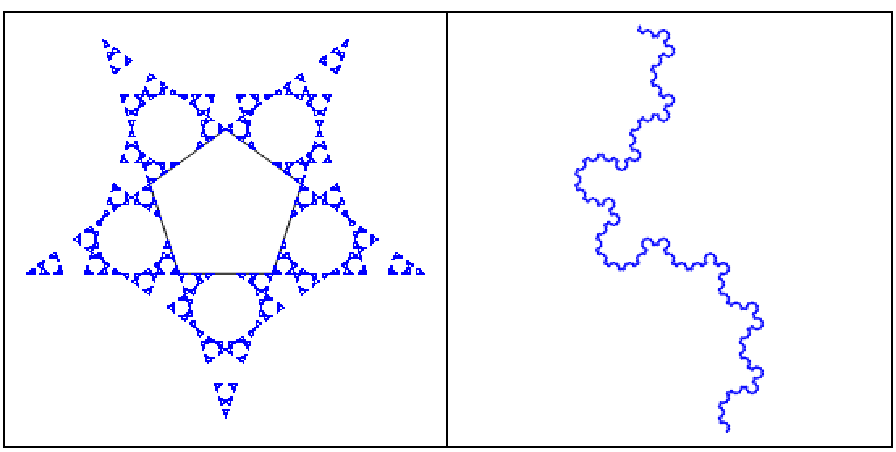
k = 2500;



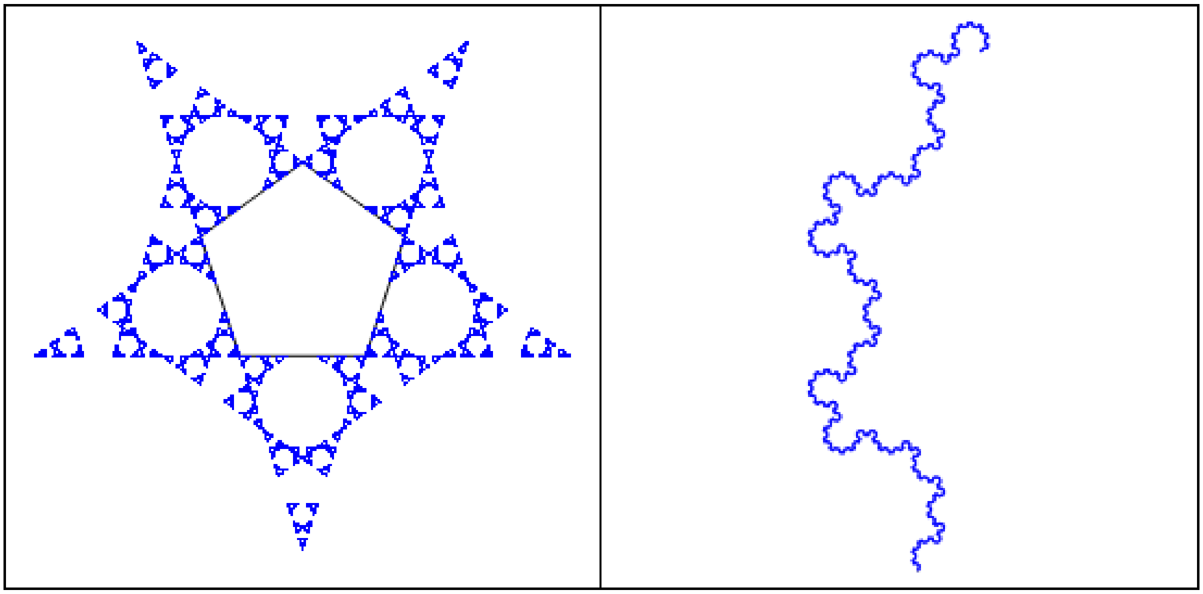
k = 25000;



k = 250000;



k = 500000;



See N10Summary for more projections.

References:

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Tabachnikov S., Dogru F. Dual Billiards, The Mathematical Intelligencer, Vol. 27, No. 4 (2005) 18-25