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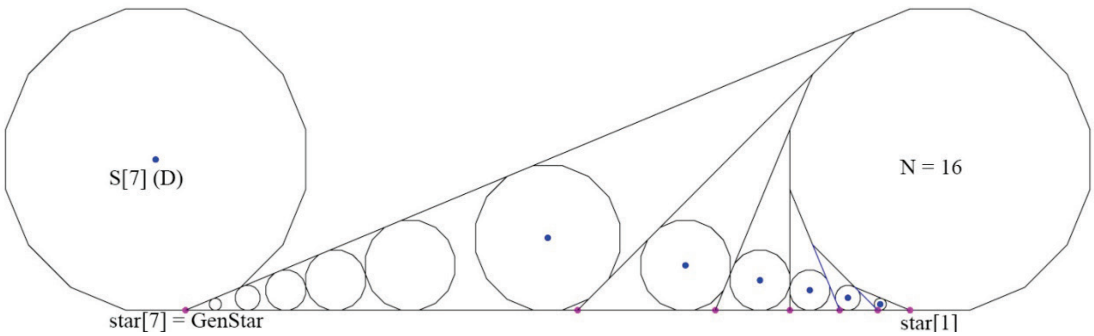
Summary of dynamics of the regular hexadecagon: N = 16

N = 16 is the first non-trivial member of the N = 2^k family. It is not clear how the members of this family are related dynamically, but geometrically they are closely related since each one is a 'bisection' of the previous. This means they form a sequence of nested factor graphs and it is no surprise that S[4] is 'mutated' for N = 16 and both S[4] and S[8] are mutated for N = 32.

The algebraic complexity grows exponentially since EulerPhi[2^k]/2 = 2^{k-2}. Therefore N = 16 has 'quartic' complexity – along with N = 15 and N = 20. In [LKV] the authors note that algebraic analysis in the quartic case appears to involve "great computational difficulties".



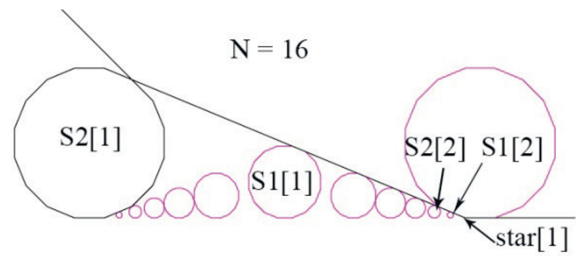
Below are the periods of the First Family where S[6] = LS[6] occupies the central position. Note that it is period 8 because of decomposition. S[2], S[4] and S[6] also have shortened periods, but S[4] is the only 'mutated' tile. It consists of two interwoven squares at slightly different radii.

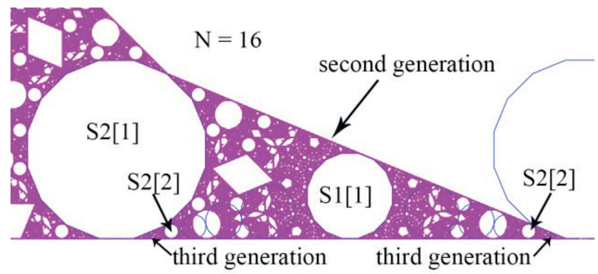
Tile	S[1]	S[2]	S[3]	S[4]	S[5]	S[6]	LS[1]	LS[2]	LS[3]	LS[4]	LS[5]	LS[6]
Period	16	8	16	4	16	8	96	40	64	12	32	8

Using the 2kN Lemma of the Digital Filter map, these orbits can be united as follows. Each S[k] and LS[k] form a combined count which can be found using the Df periods of the S[k]. This is shown below.

Tiles	S[1]& LS[1]	S[2]& LS[2]	S[3]& LS[3]	S[4]& LS[4]	S[5]& LS[5]	S[6]& LS[6]
Count	2*7*8	2*6*8	2*5*8	2*4*8	2*3*8	2*8

In twice-even cases. the canonical tiles are all N-gons – so it appears that the M-D distinction does not exist. However the notion of 'generations' for N-odd or twice-odd, still exists. When N is odd, each generation (if it exists) is presided over by an M[k],D[k] pair playing the roles of matriarch and patriarch. These canonical M[k] and D[k] tiles form on the edges and vertices of the previous D[k-1], so they are always step-1 and step-2 respectively. N = 16 preserves this step-2 vs. step-1 dichotomy, so it would not be 'politicially incorrect' to associate S[1] with M[1] and S[2] with D[1]. These two tiles are shown on the left along with a magenta virtual First Family for S2[1].

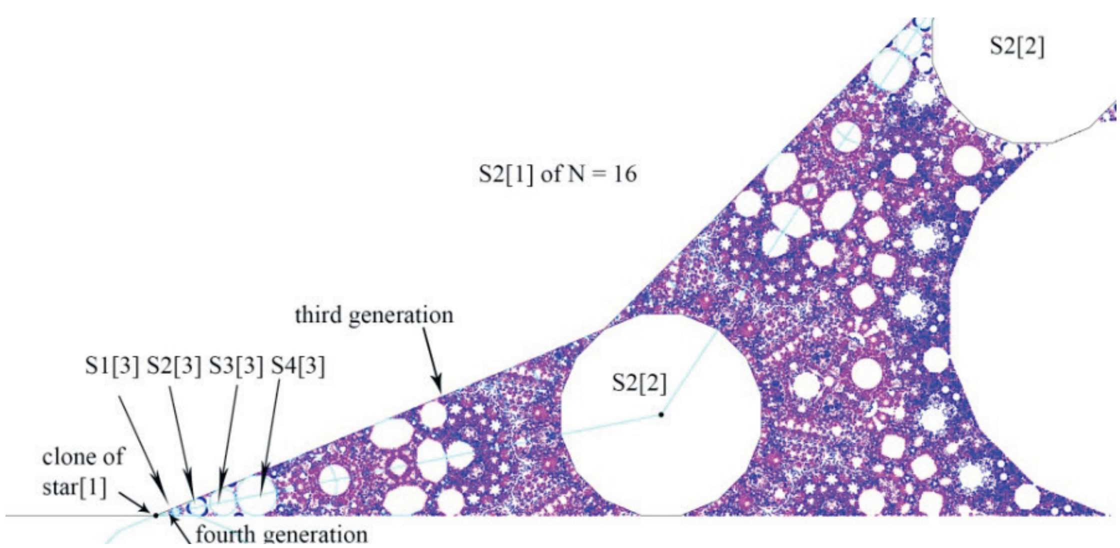




Note that S1[1] is a step-6 of S2[1] so it is natural to associate this tile with M[1]. However the web plot on the right makes it clear that S1[2] does not exist, so there is no 'M[2]]'. Since S2[2] does exist at star[1] and at the foot of S2[1], either tile can play the role of D[2]. We choose to study the foot of S2[1] because it provides valuable information about the local dynamics – which is often 'hidden' at star[1] and at GenStar.

This region is enlarged below – note that the symmetry now is with respect to S2[2]. We have reproduced the virtual First Families of S2[2] to show the perfect match with four members of the third generation- including S1[3] which is the surrogate M[3]. The first 10 S2[k]'s in this sequence have periods 8, 32, 456, 2464, 20872, 110368, 974664, 5165216, 45423368 and 240668192 which gives ratios of about 4.66 and 8.8.

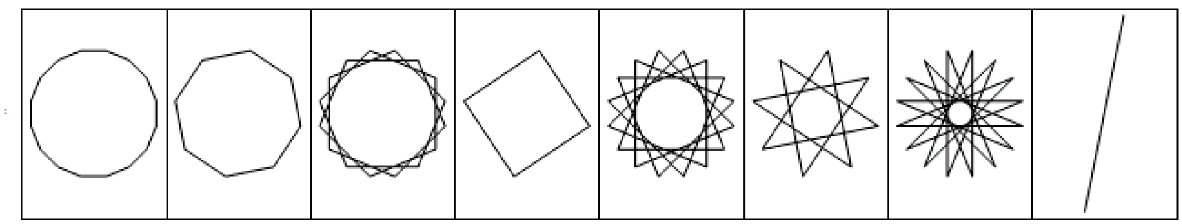
As with N = 7, the ratios alternate high-low within the even and odd sequences. N=20 may also support generations – but they are complicated by the fact that S2[1] is a mutated decagon.



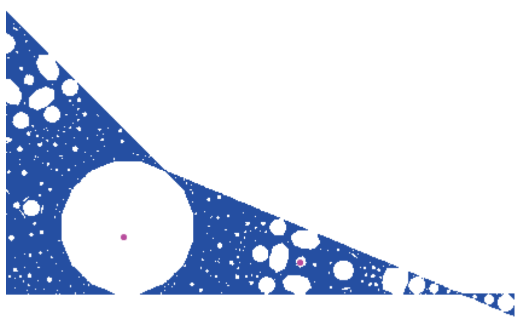
Projections

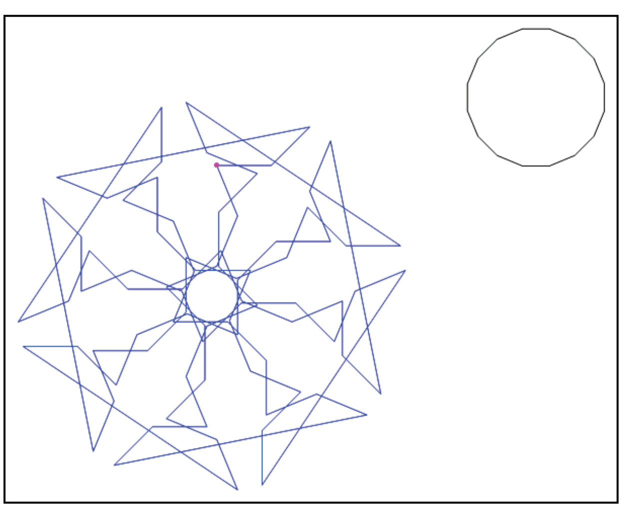
EulerPhi[16] = 8 so there are 4 non-redundant projections but our convention is to look at all 7 . Below are the vertex assignments for the projections, showing 3, 5 and 7 as the relatively prime projections.

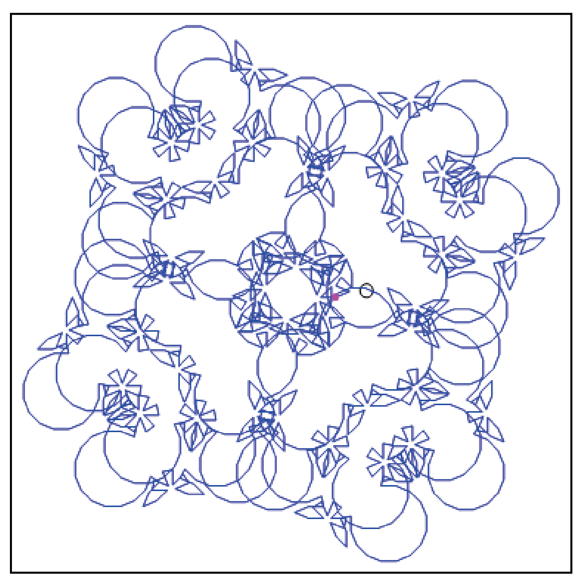
GraphicsGrid[{ Table[Graphics[poly[Wc[[k]]]],{k,1,8}]},Frame->All]



Example 1: The magenta dot in D[2] below has period 176 and the second dot, where M[2] should be, has period 2952. The projections have periods which are half of these. Below is the P7 projection for D[2] and the P5 projection for 'M[2]'.







The following are samples of projections from each of the three invariant regions:

Example 2: The Inner ring: q1= {-.67013678100000001233, -.75300001627788811};

Ind = IND[q1, 50000]; k= 250; (*plot the first 250 iterations*)

```
GraphicsGrid[{{Graphics[{poly[Mom],Blue,Point[PIM[q1,k,1]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,2]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,3]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,4]]}]},
{Graphics[{poly[M],Blue,Line[PIM[q1,k,5]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,6]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,7]]}]}},Frame->All]
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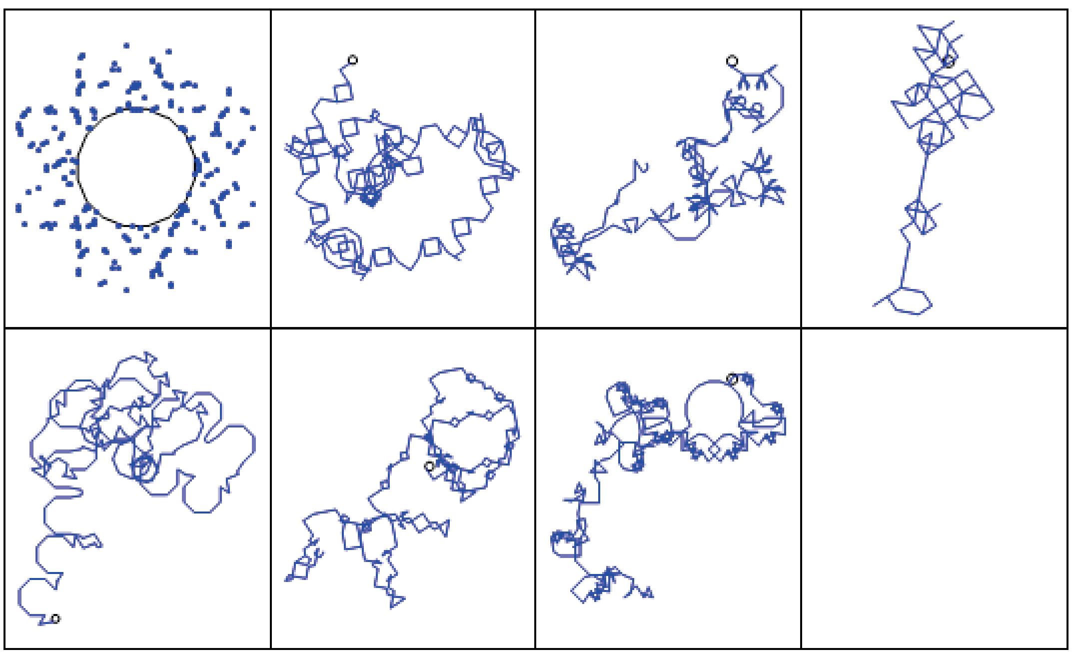
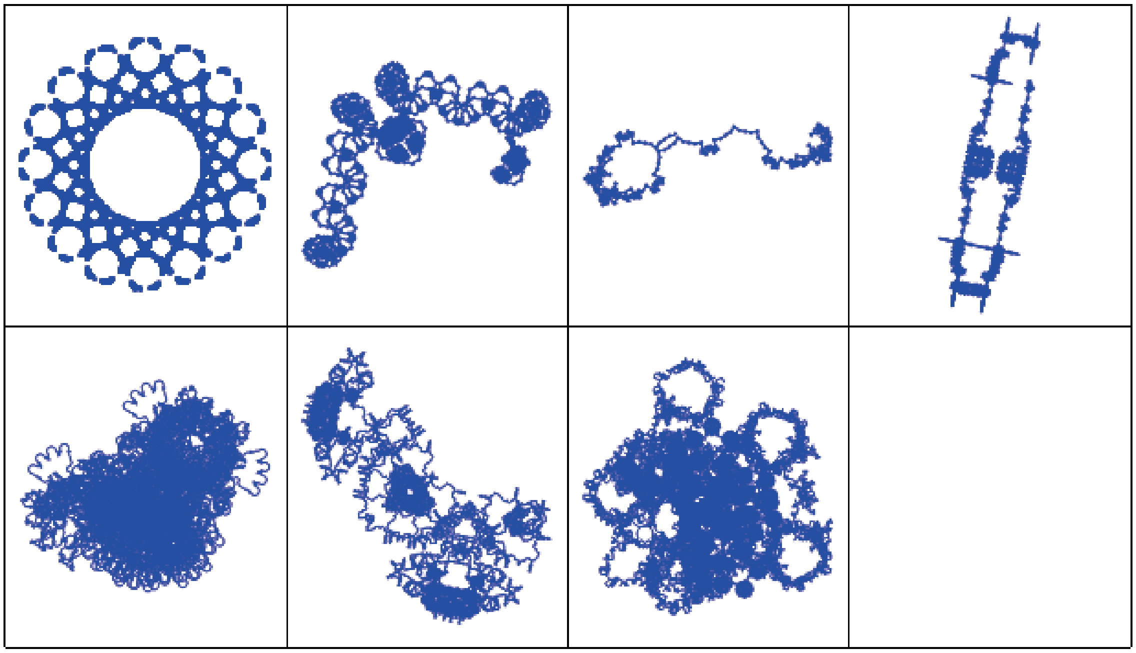
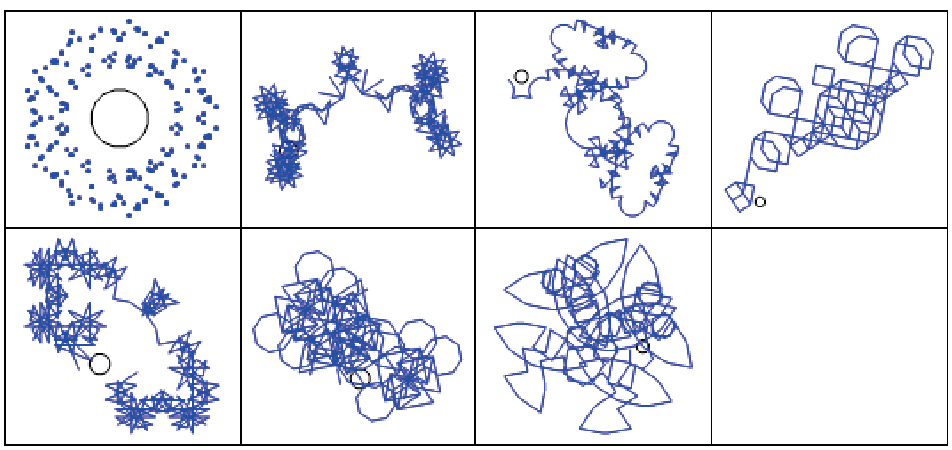


Now **k = 20000**

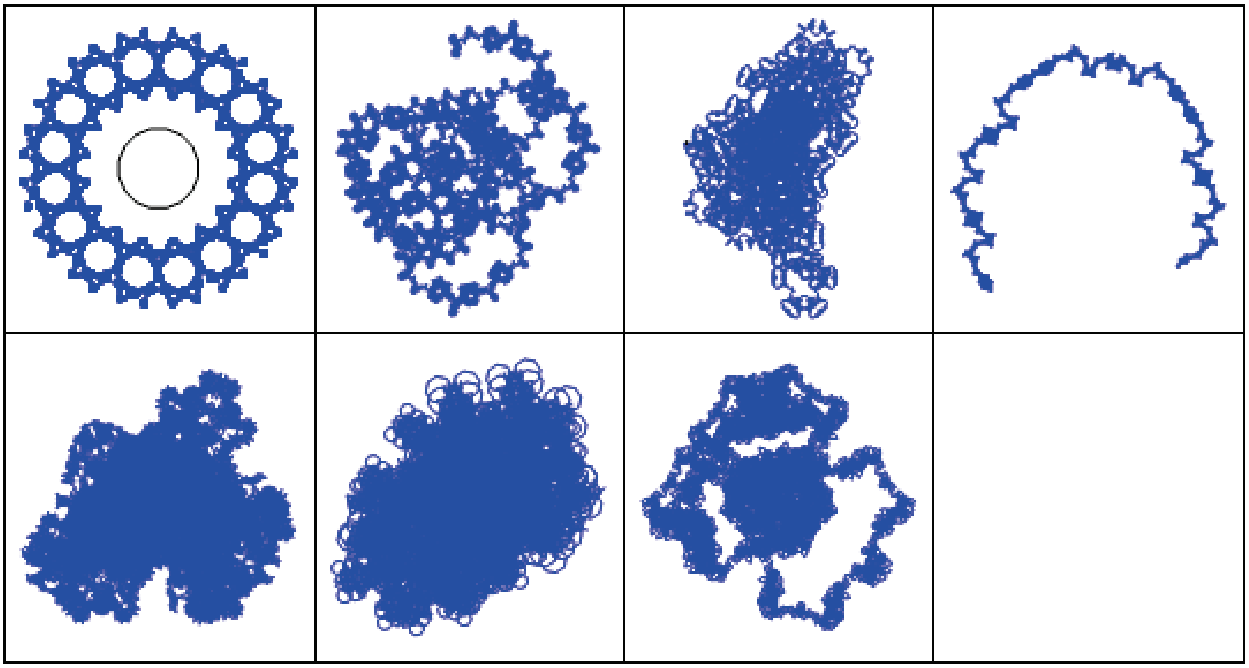


Example 3: The middle ring: q1= {-2.08112357199000010110, -.53262317191122200001}

k = 250

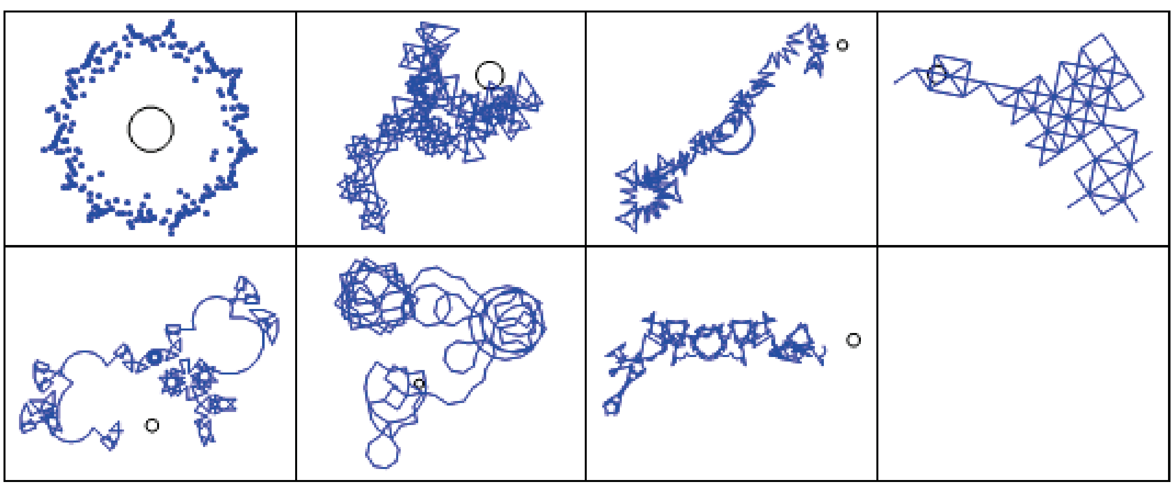


k = 20000. Note the increase in density as we move outwards.

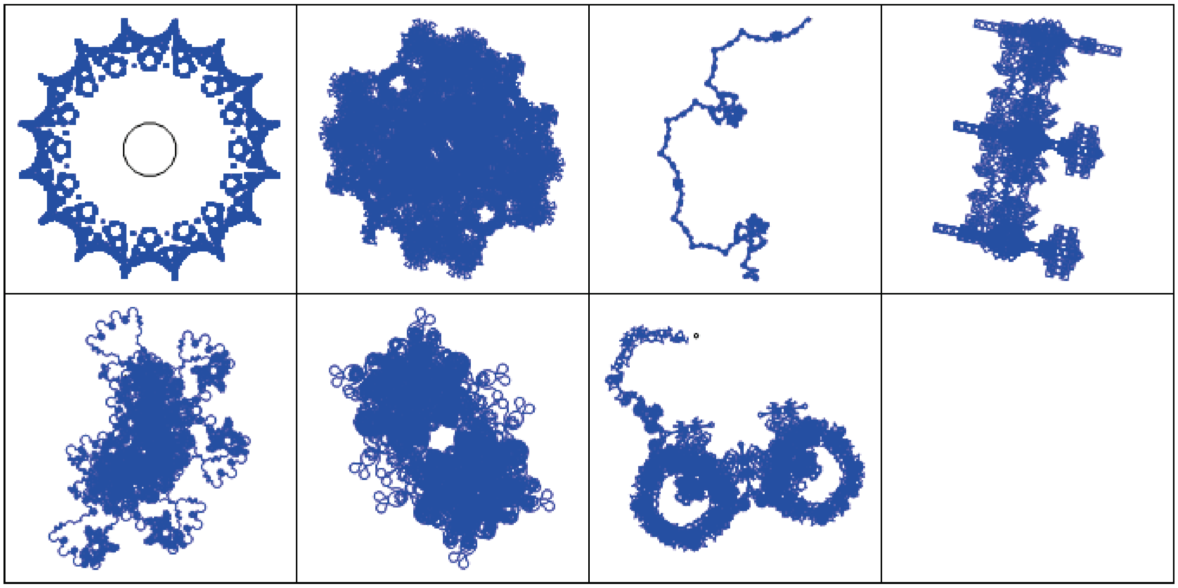


Example 4: The outer ring: q1= {-4.9012667100111113, -.973701771231000001}

k = 250;

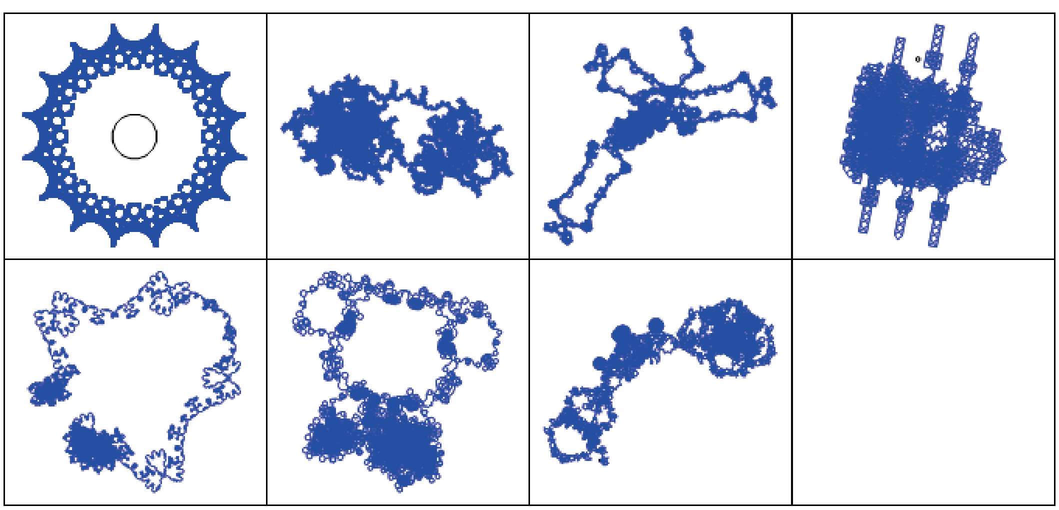


k = 20000. Yes, P7 is a bicycle and P5 is waving frantically.

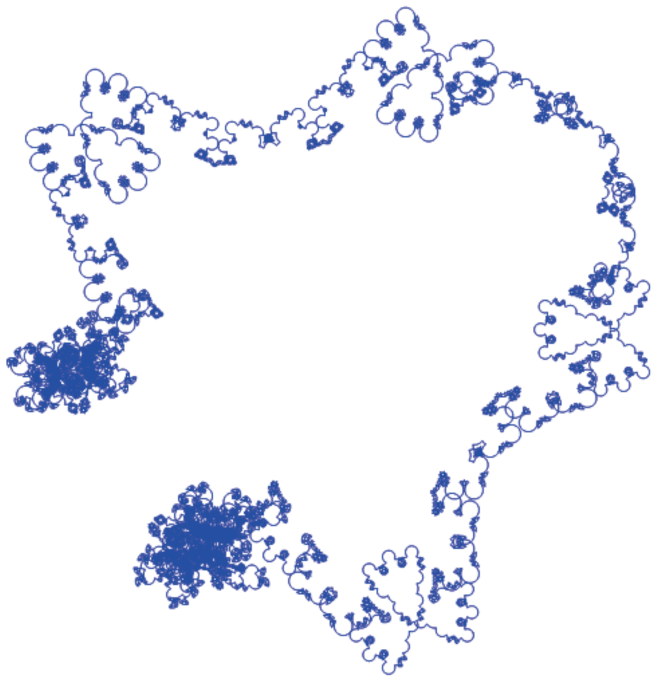


This last point has been a 'work-horse' because it is remarkably dense in generation 2 and beyond. (See the earlier plots.) Since we have tracked the orbit for billions of iterations we can skip ahead in the projections using an advanced initial point.

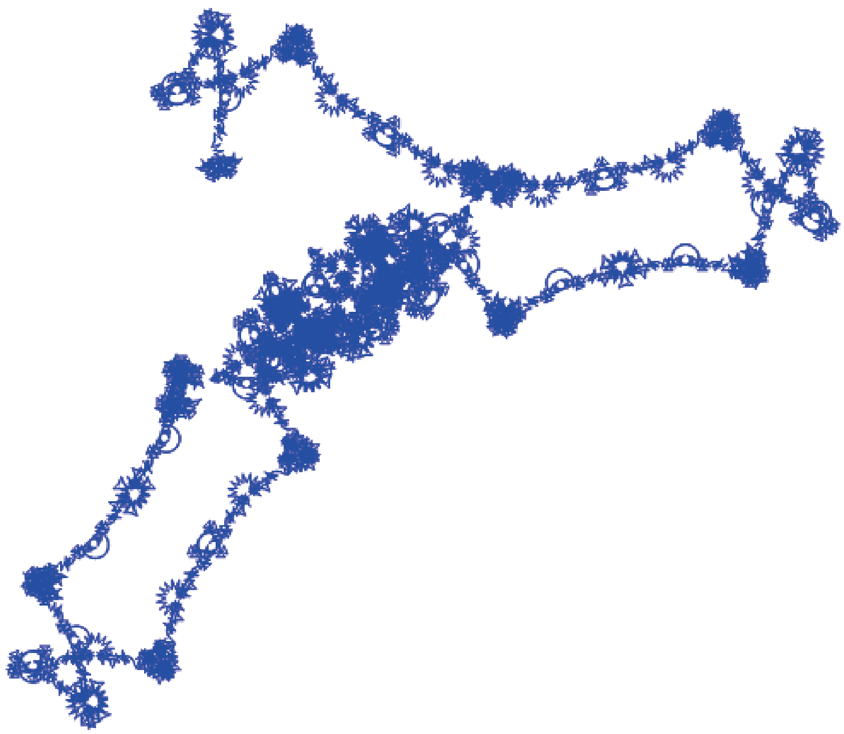
Example 5: q1 = {3.6910131180033360897388,0.0929005697712447 13735} which is an advance of 65 million iterations from q1 in Example 4 above.
k = 20000



Below is an enlargement of P5 showing just the first 15,000 points. The hands which had four fingers now have three.**Graphics[{Blue,Line[PIM[q1,15000,5]]}]**

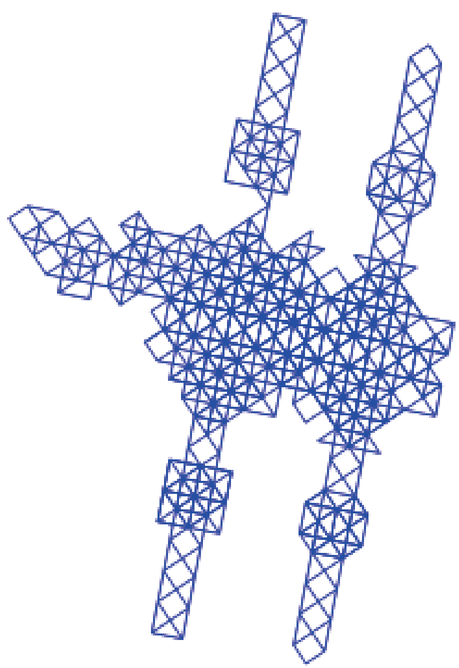


All the projections share the same symmetry. Below is P3 for these same parameters.
Graphics[{Blue, Line[PIM[q1, 15000, 3]]}]



As expected, P4 is an 'erector set'. Below are the first 4000 points.

Graphics[{Blue,Line[PIM[q1,4000,4]]}]



References

Hughes G.H., Outer billiards on regular polygons, **arXiv:1206.5223**

[LKV]Lowenstein J. H., Kouptsov K. L. and Vivaldi . F, Recursive tiling and geometry of piecewise rotations by π/7, Nonlinearity 17 1–25 MR2039048 (2005)f:37182)