GeneticsOfPolygons.org

Lattice Polygons

Definition: A *lattice polygon* is a polygon where the vertices are elements of $\mathbb{Z}^d$ for some $d \geq 2$ (where $\mathbb{Z}$ is the set of integers)

Example1: Below are some lattice polygons in the square lattice $\mathbb{Z}^2$ including a rectangle and an octagon, which are the only two equiangular lattice polygons in $\mathbb{Z}^2$, so the square is the only regular lattice polygon. The hexagon and triangle shown here are almost equiangular and they can be made regular in a cubic lattice as shown in Example 2.

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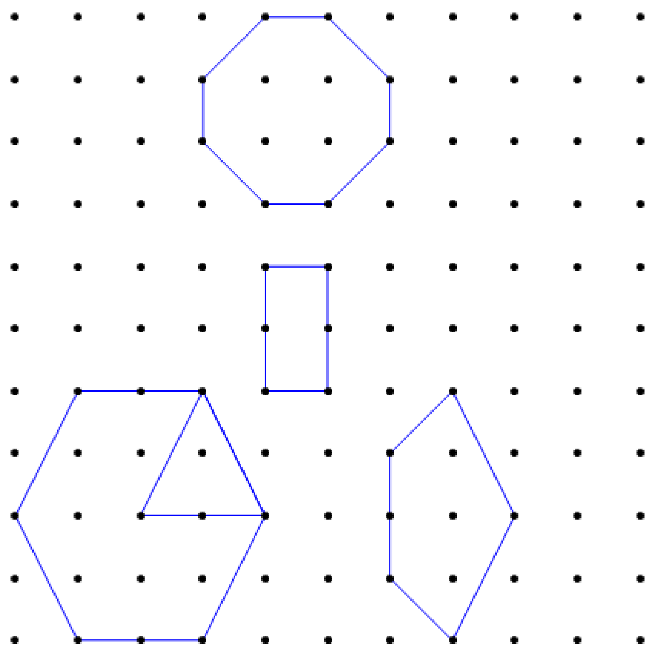
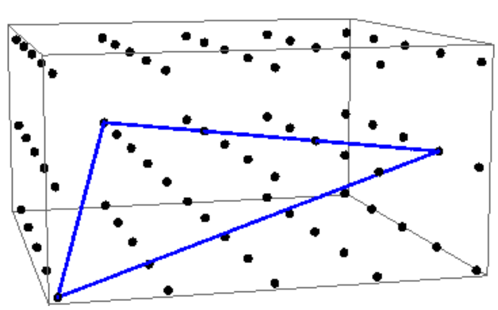
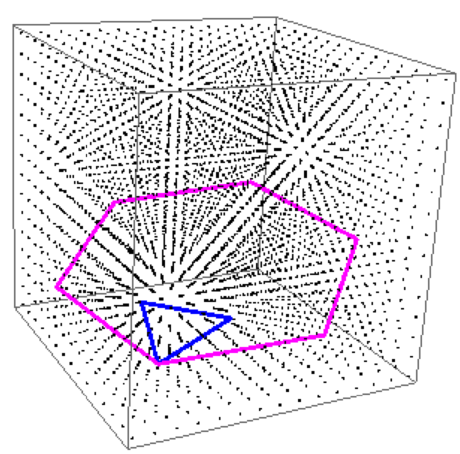


Example 2: A regular triangle and regular hexagon in $\mathbb{Z}^3$. The (counterclockwise) coordinates of the triangle are {0,0,0}, {4,1,1} & {1,4,1} and the hexagon has coordinates {0,0,0}, {7,-2,1}, {12,3,3}, {10,10,4}, {3,12,3},{-2,7,1}





Theorem:

(i) The rectangle and octagon are the only two equiangular lattice polygons in $\mathbb{Z}^2$.(Since the equiangular octagon is not equilateral, the square is the only regular lattice polygon in $\mathbb{Z}^2$.)

(ii) The square, triangle and hexagon are the only regular lattice polygons in $\mathbb{Z}^d$ for any dimension d. The square can be embedded in $\mathbb{Z}^2$ while the triangle and hexagon require $\mathbb{Z}^3$.

Both of these results are based on a result by D.H. Lehmer concerning the minimal polynomial equations which arise in the study of cyclotomic fields.

Lehmer's Lemma: For $n \geq 2$, $2\cos(2\pi/n)$ is an algebraic number of (minimal) degree $\varphi(n)/2$. (where $\varphi(n)$ is the Euler totient function)

We will discuss the proof of Lehmer's Lemma, but first we will give a proof of the Theorem above using the lemma.

Part (ii) was proven by H.E. Chrestenson of Reed College in 1963 in answer to a question by M. S. Klamkin who noted that an equilateral triangle could not be embedded in Z^2 but it could be embedded in $\mathbb{Z}^3$. Klamkin wondered if any regular polygon could be embedded in a lattice $\mathbb{Z}^d$ if the dimension d was high enough ?

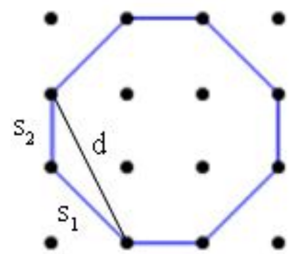
To answer this question, Chrestenson noted that if P is a regular n-gon with side s embedded in a lattice Z^d, then the law of cosines applied to s and the shortest diagonal d of the polygon gives

$$d^2 = 2s^2 - 2s^2 Cos(\pi - 2\pi / n) = 2s^2 + s^2(2Cos(2\pi / n)$$

Since d^2 and s^2 are integers, $2\cos(2\pi/n)$ must be rational which implies that $\varphi(n) = 2$ and the only possible cases are 3,4 or 6. Chrestenson then used the triangle embedding given in Example 2 earlier to obtain the hexagon embedding and show that $\mathbb{Z}^3$ is sufficient for the triangle and the haxagon (and the square), but no dimension would suffice for the rest of the regular polygons.

Part (i) was proven by Dean Hoffman and it appeared in "Mathematical Gems' by R. Honsberger in 1982. Lemma: A convex lattice n-gon is equiangular iff n = 4 or n = 8.

We will present the somewhat 'streamlined' proof by P.R Scott. The proof follows the same lines as the Chrestenson's proof. Suppose P is an equilateral lattice n-gon and 2 adjacent sides have length s_1 and s_2. Let d be the third side of the triangle so s_1^2, s_2^2 and d^2 are integers.



The included interior angle is (n-2)π/n and cos((n-2)π/n) = -cos(2π/n) so the law of cosines gives $d^2 = s_1^2 + s_2^2 + 2s_1s_2\cos(2\pi/n)$. Therefore cos(2π/n) has the form $a/\sqrt{b}$ with a and b integers. This implies that 2cos(2π/n) is an algebraic integer of degree at most 2, so φ(n) ≤ 4. The only posible values for n are 3,4,5,6,8,10 and 12. To eliminate all but 4 and 8 note that for an integer lattice, Tan(2π/n) is either rational or infinite and n = 4 is infinite while n = 8 is 1. The rest are irrational.

It is natural to ask if anything changes if we allow affinely regular polygons, and the answer is no. In their study of discrete tomography, Gardner and Gritzmann (1997) have shown that the affinely regular lattice polygons are exactly the affinely regular triangles, parallelograms and hexagons.

Minimal Polynomials (see Construction of Regular Polygons.pdf**)**

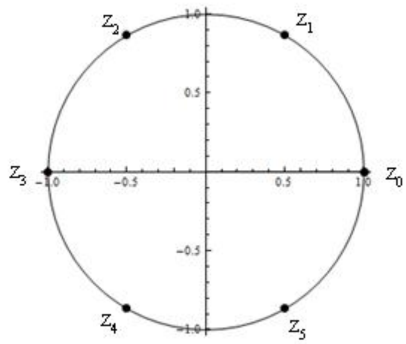
Definition: An *algebraic number* is a number that is a root of a non-zero polynomial in one variable with rational coefficients.

Definition: The *minimal polynomial* of an algebraic number z is the unique irreducible polynomial p(x) with rational coefficients and leading coefficient 1, such that p(z) = 0.

For every value of n, the primitive roots of the cyclotomic equation $z^n = 1$ all have the same minimal polynomial which is called the nth cyclotomic polynomial Φ(n). (The primitive roots z of the cyclotomic equation are those where $z^n = 1$ and n is the smallest positive integer with this property.)

Definition: The nth cyclotomic polynomial is $\Phi(n) = \prod_{k=1}^{n}(x - z_k)$ (z_k primitive)
(It is not trivial to prove that this polynomial is always irreducible. For a proof see van der Wearden's Algebra, section 8.4.In Mathematica type **Cyclotomic[n,x]**.

Example: For n = 6, z_1 and z_5 are the only primitive roots. The number of primitive roots is φ(n) and this is always the degree of the cyclotomic polynomial.



So the minimal cyclotomic polynomial $\Phi(6)$ is $(x - z_1)(x - z_5) = x^2 - x + 1$ since $z_1 + z_5 = 2\cos(2\pi/6) = 1$ and $z_1 z_5 = 1$.

For any cyclotomic polynomial $\Phi(n)$, the primitive root $z = \cos(2\pi/n) + i\sin(2\pi/n)$ can always be paired with its complex conjugate $\bar{z}$ as in the example above, so that $z + \bar{z} = 2\cos(2\pi/n)$. Since z is a root of $\Phi(n)$ and z is a quadratic extension of $\cos(2\pi/n)$, it make sense that the degree of $\cos(2\pi/n)$ should be half the degree of $\Phi(n)$. When n is prime the pairing is perfect so the degree of $\sin(2\pi/n)$ will be $\varphi(n)$.

In general the relationship is as follows (from Paulo Ribenboim,Algebraic Numbers)

(i) The degree($\cos(2\pi/n) = \varphi(n)/2$ (Lehmer's Lemma)

(ii) If $n \neq 4$ and $n = 2^r m$ for m odd, $\deg(\sin(2\pi/n)) = \left\{ \begin{array}{l} \varphi(n) \text{ if r = 0 or 1} \\ \frac{1}{4}\varphi(n) \text{ if r = 2} \\ \frac{1}{2}\varphi(n) \text{ if } r \geq 3 \end{array} \right\}$

For n = 5, $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$ is the minimal polynomial for the four primitive roots $z = \cos(2\pi k/5) + i \, \mathrm{Sin}(2\pi k/5)$ $(k > 0)$ but individually $\cos(2\pi/5)$ has a minimal polynomial of degree 2 (namely $4x^2 + 2x - 1$), while $\sin(2\pi/5)$ is degree 4 ($16x^4 - 20x^2 + 5$).

All regular polygons must have coordinates with algebraic form, because they are roots of the algebraic equation $\zeta^n = 1$ and the Fundamental Theorem of Algebra says that there must be n (algebraic) roots in C. There are formulas for the minimal $\cos(2\pi/n)$ and $\sin(2\pi/n)$ polynomials. In the April, 2004 Mathematics Magazine, Scott Breslin and Velerio De Angelis give a derivation of these formulas when n is prime, using only Eisenstein's criteria for irreducibility. These polynomials are closely related to the Chebyshev polynomials of the second kind. Mathematica will do the general case by typing **MinimalPolynomial[Cos[2Pi/n]]** or **MinimalPolynomial[Sin[2Pi/n]]**

For example with the regular hexagon as shown above, φ(6) = 2, so the minimal equation for cos(2π/6) is degree 1, which means cos(2π/6) is rational (in fact equal to 1/2). On the other hand, the minimal equation for sin(2π/6) is degree 2 and $\sqrt{3}/2$ is the best possible. (This explains why the regular hexagon is not a lattice polygon in $\mathbb{Z}^2$.)

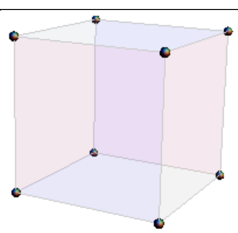
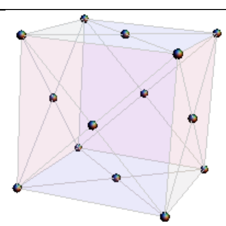
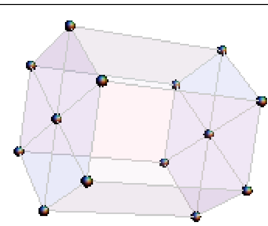
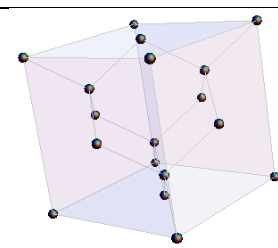
For n = 12, φ(n) = 4 and the minimal polynomial for cos(2π/12) is $4x^2 - 3$ while the minimal polynomial for sin(2π/12) is just 2x – 1. For n = 24, φ(n) = 8 and the minimal polynomila for cos(2π/24) is $16x^4 - 16x^2 + 1$ and this is also the minimal polynomial for sin(2π/24).

For n = 5, cos(2π/5) is just degree 2 but sin(2π/5) is minimal degree 4. Therefore cos(2π/5) = ($\sqrt{5}$ -1)/4 but sin(2π/5) = $\sqrt{5/8 + \sqrt{5/8}}$. In terms of constructability, this is not an issue because constructing one of these is the same as finding both. Virtually all the construction methods for the regular pentagon use the cos formula.)

Since φ(n) ≤ 2 for n =3, 4 and 6 these are the only polygons where cos(2π/n) is rational and n = 4 is the only case where sin(2π/n) is also rational. For n = 5, 8 and 12 , φ(n) = 4, so the minimal cosine polynomial is degree 2 and this explains why these are constructible. For most prime beyond n = 5 there is a problem with constructability but at n = 17, φ(n) = 16 and this yields a path to constructability by the same methods used to attack n = 5.

Generalized Lattices

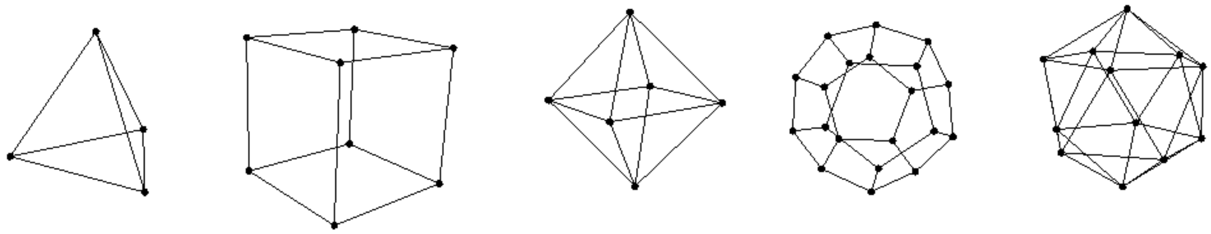
In the definition of a lattice polygon we assumed that the polygon is embedded in Z^d but the base lattice could be any regularly-spaced array of points in $\mathbb{Z}^d$ or $\mathbb{R}^d$. Below are some examples from the Mathematica data-base of Lattices.

			
Z^3 (Cubic lattice)	Face-centered cubic	Hexagonal	Tetrahedral packing

There are important applications of geometric lattice theory to tomography which attempts to determine information about 3 dimensional objects based on planar 'slices'. These scans are typically extracted from X-rays (CT) or positron emissions (PET). Applications include biology, geophysics, archeology, astrophysics, chemistry and nanotechnology. In the study of quasicrystals there are classes of cyclotomic model sets (algebraic Delauney sets) which serve as models for tilings and corresponding cross -sections.

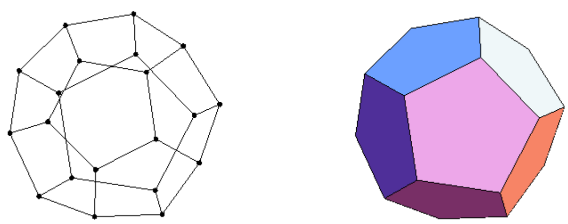
Even in higher dimensions the regular polygons and their affine images, continue to play an important role as building blocks of polyhedra. The Greeks knew that there are only 5 regular polyhedra (all faces regular and each vertex has same number of faces). These are also known as the Platonic Polyhedra.

PolyhedronData["Platonic"] yields { Tetrahedron , Cube, Octahedron, Dodecahedron, Icosahedron }



These plots are in "VertexForm" with the vertices shown explicitly. For example, to plot the Dodecahedron: **V1 = PolyhedronData["Dodecahedron", "VertexCoordinates"]; (*the vertices*)**

Graphics3D[{PolyhedronData["Dodecagon", "Edges"], AbsolutePointSize[6.0], Point[V1]}, Boxed->False] (* on the right, "Edges" is changed to 'Faces" *)

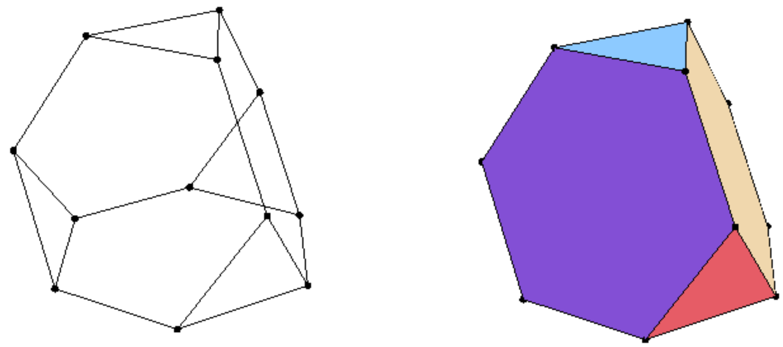


In 1959, E. Ehrhart showed that only the tetrahedron, cube and octahedron are lattice polyhedron.

PolyhedronData["Archimedean"] yields the 13 semi-regular Archimedian polyhedra:

{Cuboctahedron, GreatRhombicosidodecahedron, GreatRhombicuboctahedron, Icosidodecahedron, SmallRhombicosidodecahedron, SmallRhombicuboctahedron, SnubCube, SnubDodecahedron, TruncatedCube, TruncatedDodecahedron,TruncatedIcosahedron, TruncatedOctahedron, TruncatedTetrahedron}

These polyhedron are made up of regular faces but the faces may be mixed. For example the TruncatedTetrahedron shown below has a mixture of regular triangles and hexagons.



P. R. Scott showed that of these 13 semi-regular polyhedra, only the truncated tetrahedron (above), the truncated octahedron and the cuboctahedron are lattice polyhedra.

Bibliography

Breslin, S., DeAngekis, V.,The Minimal Polynomials of sin(2π/p) and cos(2π/p), Mathematics Magazine, April 2004

Klamkin, M.S, Chrestenson, H.E.Polygon Imbedded in a Lattice, Advanced Problems and Solutions, American Mathematical Monthly, Vol 70, No.4, April 1963, pp 447-448