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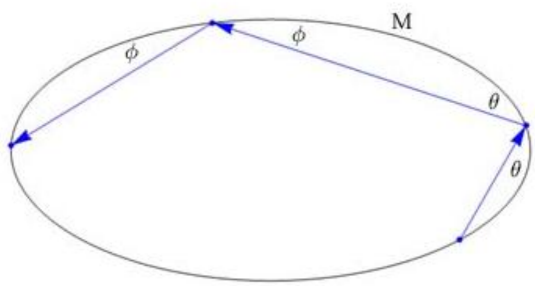
Inner billiards

Inner Billiards

The mathematical study of inner billiards began with George David Birkhoff who defined the 'billiard ball problem' in his 1927 book "Dynamical Systems".The related study of optics and the reflective properties of surfaces is thousands of years old and in the 18th century there were important connections made between optics and the study of mechanics. Birkhoff's work helped to establish a relationship between geometry, topology and statistical physics. This relationship played a critical part in his 1931 ergotic theorem which addressed the fundamental issues of statistical mechanics.

Inner billiards is usually played inside a closed 'convex' curve which is typically smooth.The ball is a point mass moving with constant momentum. The collisions with the boundary are assumed to be elastic so the angle of incidence is equal to the angle of reflection.This is called a*specular*(mirror like) reflection. At the collision point, there is no change in the tangential component of momentum, and there is an instantaneous reversal of the normal component of momentum.



The dynamics of inner billiards tends to be more complex than outer billiards. For example two polygons which are affinely equivalent have essentially the same dynamics relative to the Tangent Map but this is no longer true for inner billiards. The smaller symmetry group typically implies a higher degree of complexity. But for smooth curves with positive curvature, the dynamics of inner billiards are similar to outer billiards.

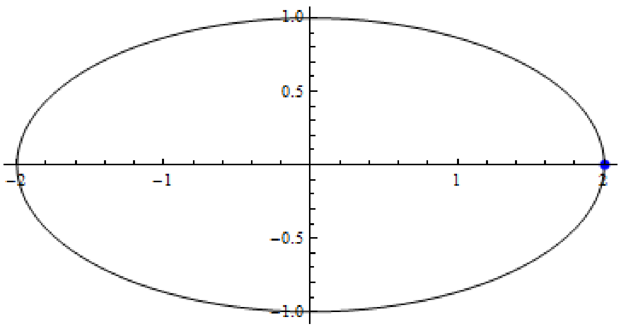
Example: An elliptical table: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or in parametric form $\{aCos[u], bSin[u]\}, \{u, 0, 2\pi\}$

For a discrete approximation with 200 vertices: **M = Table[{aCos[t], bSin[t]},{t,0,2 Pi,2 Pi/200];**

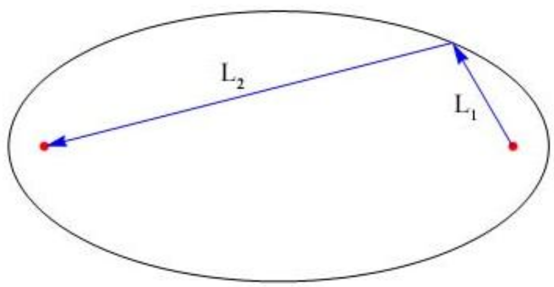
This approximation will not be used for the inner billiards map, but it will be used to compare the outer billiards map (Tangent Map) to the inner billiards map. For the purposes of the Tangent Map software, we will need a clockwise orientation so

M= Reverse[M].(*All the graphics below are based on the notebook InnerBilliards.nb*)

a = 2; b = 1; Graphics[{poly[M], Blue, AbsolutePointSize[6.0],Point[M[[1]]] }, Axes->True]

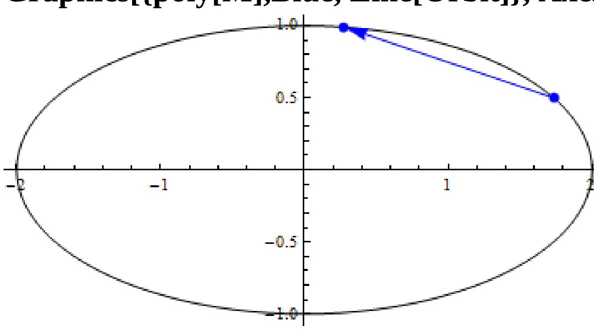


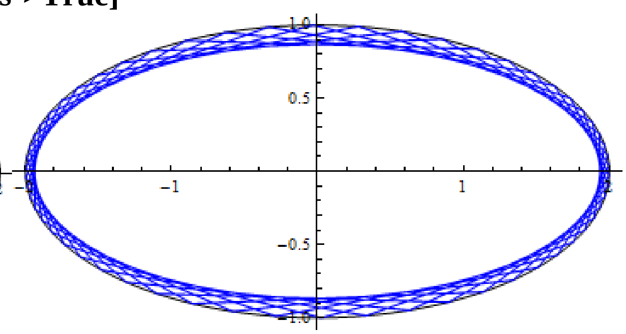
The two foci of this ellipse are at $\left\{-\sqrt{3},0\right\}$ and $\left\{\sqrt{3},0\right\}$ as shown below. The sum $L_1 + L_2$ is constant and this property can be used to define the ellipse.



Any billiards trajectory that passes through one focus point, must map to the other. Most billiards orbits of an ellipse operate inside or outside the two foci. A typical orbit inside the foci will generate a hyperbola and a typical orbit outside will generate a 'confocal' ellipse – that is an inscribed ellipse with the same foci. Below is an example.

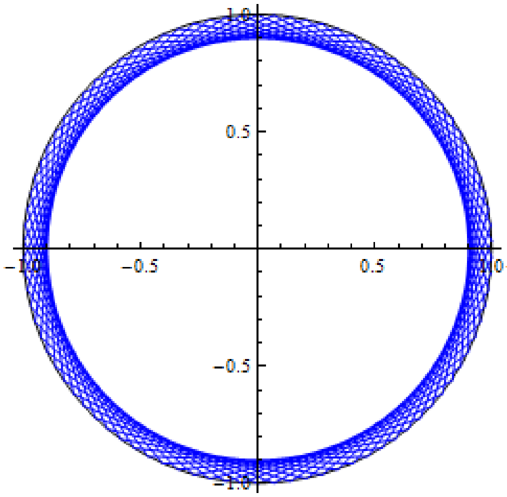
angles= {30Degree,82Degree}; These two points on the ellipse define the initial vector as shown on the left below On the right are the first 150 points in this orbit:
Do[AppendTo[angles,newpoint[angles[[{-2,-1}]],{2,1}]],{i,150}];
Orbit =Table[Ellipse[{2,1},angles[[i]]],{i,Length[angles]}];(*newpoint generates the next angle and Ellipse plots it*)
Graphics[{poly[M],Blue, Line[Orbit]}, Axes->True]

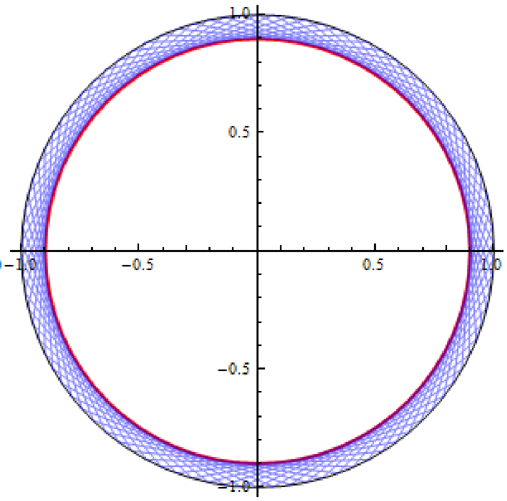




This orbit forms a 'caustic' which in this case is a confocal ellipse.Below is this same orbit, inside a circle.The 'winding number' is fixed at θ = 52 degrees, and **LCM[52,360]** = 4680 so it will repeat after

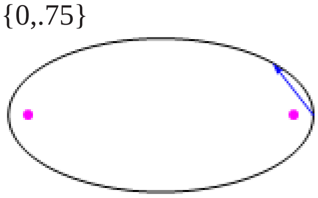
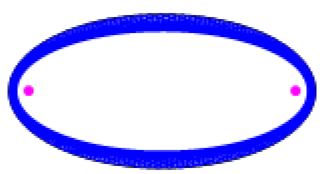
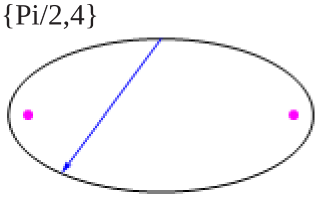
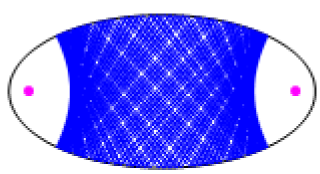
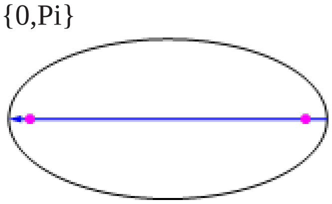
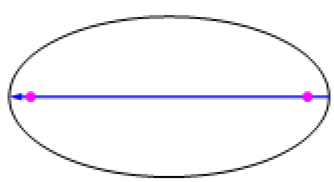
90 iterations. The caustic will half the rotation angle so the radius of the inscribed circle is **Cos[26Degree]** ≈ .89879 as shown on the right in magenta.

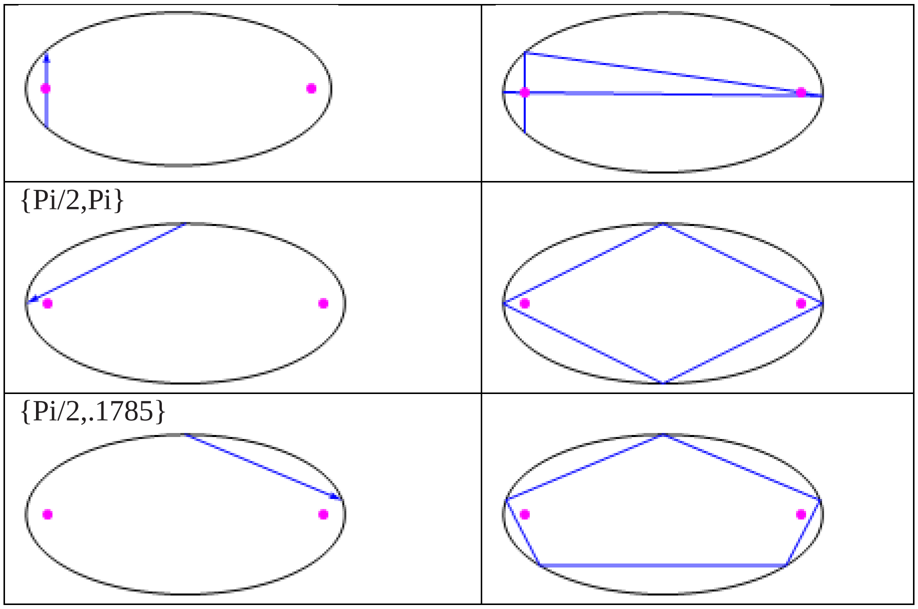




When the winding number θ not a rational multiple of Pi, the orbit of the inner billiards map will be uniformly dense in the circle. This corresponds to an ‘invariant’ torus in the Hamiltonian formulation. Note that in the plots above we show the lines, but for the inner billiards map the points themselves always lie on the ellipse or circle.

Returning to the elliptic case, below are some typical orbits

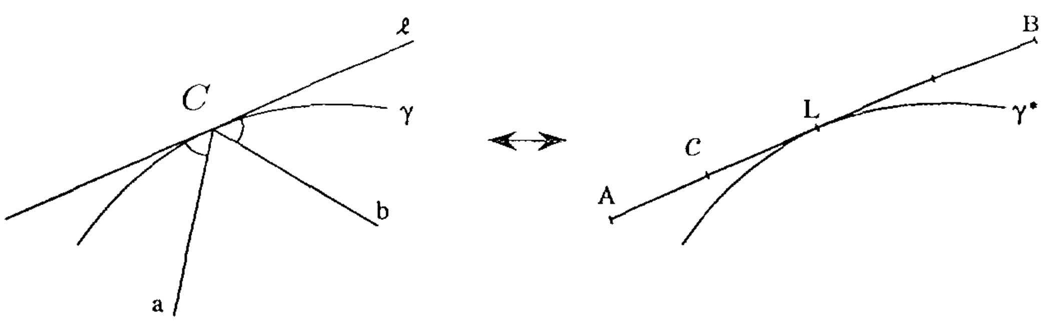
Initial Vector	Orbit
{0,.75} 	
{Pi/2,4} 	
{0,Pi} 	
$\left\{\text{Pi} + \text{ArcCos}\left[\sqrt{3}/2\right], \text{ArcCos}\left[-\sqrt{3}/2\right]\right\}$	



Periodic trajectories of the billiards map are inscribed polygons with minimal perimeter and periodic orbits of a dual billiard map are circumscribed polygons with maximal area. This is a manifestation of the projective duality on the unit sphere between lines and points.
In this duality, a minimal perimeter yields a dual polygon with maximal sum of angles and the Gauss-Bonnet Theorem guarantees that it has maximal area. (This spherical duality does not extend to the plane so it does not apply to polygonal billiards.)

Lemma (S. Tabachnikov) : For smooth convex curves γ the dual and direct billiard problems are projective dual

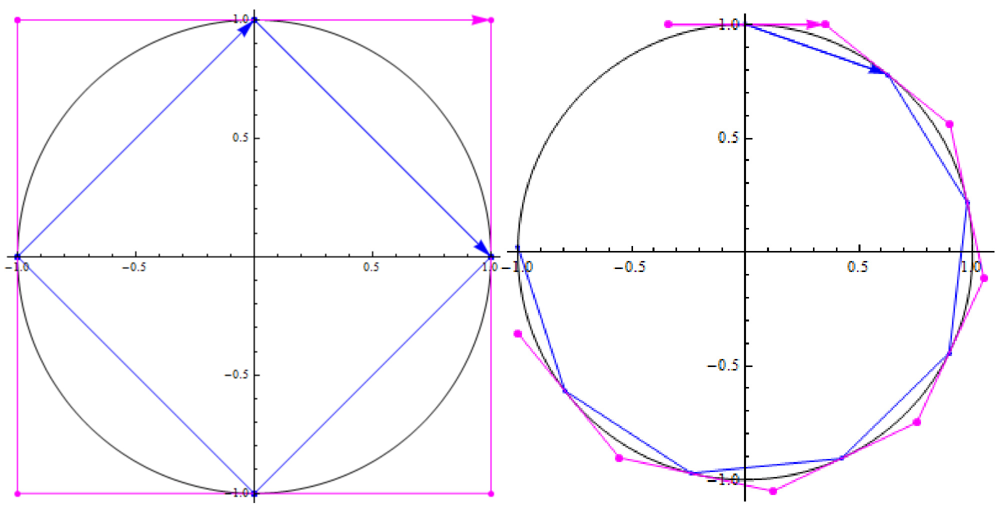
This is illustrated below with a billiard curve γ and its projective dual γ* which consists of points dual to tangent lines to γ.



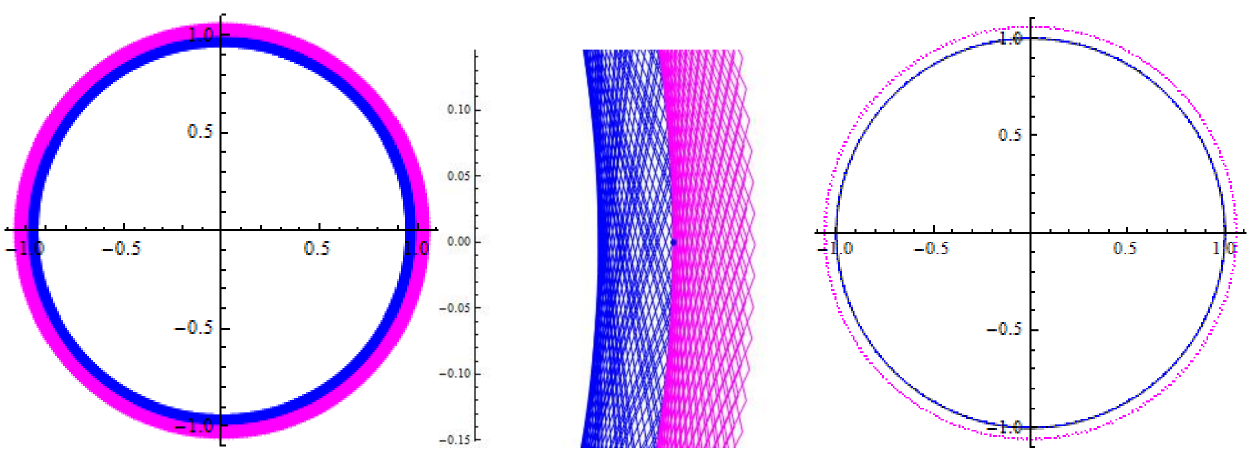
The ray *b* is the reflection of *a* under the (inner) billiard map. The line l is the tangent line to γ at the point C. In the duality, the line l corresponds to the point L and the line a corresponds to the point A. On the projective sphere, the distance between points is proportional to the angle between the lines. The angles correspond to the distances between A and L and B and L. These are equal iff the map is the tangent map.

Example: For a circle, the outer billiards map (Tangent Map) and the inner billiards map will shadow one another when the Tangent Map orbit is constant step. Below are two examples. Step 50 for N = 200 on the left and Step 30 for N = 281 on the right. The first is period 4 and the second is period 562 (period doubling). The Tangent Map is magenta and the inner billiards map is blue. We use the first iteration of the Tangent map to determine the initial vector for the inner billiards map.

q1 = {-1,1}   (N = 200)   q1 = {-.34,1} (N = 281)



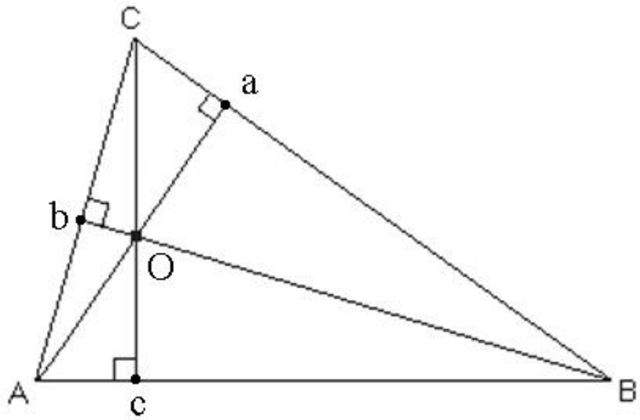
Below are the full orbits for N= 281, first as lines and then as points.



Polygonal Inner Billiards

As indicated above, the projective duality between inner and outer billiards breaks down for polygonal billiards so the affine similarity no longer exists. For example in dual billiards, all triangles have the ‘same’ trivial dynamics, but for inner billiards the dynamics of triangles is far from trivial. It is not even known whether every triangle has periodic orbits. If the triangle is acute, the construction below due to Giulo Fagnano (1682-1766) will generate a period 3 orbit {a,b,c}.

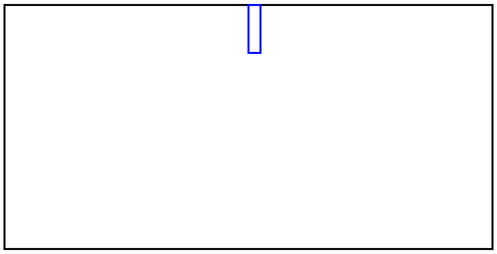
The triple intersection point O is called the orthocenter. It always exists but it is only inside the triangle when the triangle is acute. The triangle {a,b,c} is called the 'orthic' triangle of ABC. Fagnano wanted to find an inscribed triangle with smallest possible perimeter and he used the newly developed calculus to show that the orthic triangle had this property.

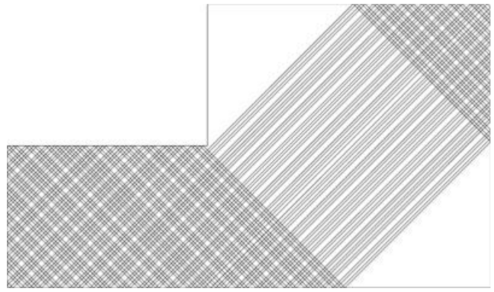


For right triangles, the orthocenter is on an edge and for obtuse triangles it is outside, so in both cases this construction fails. For right triangles there are other constructions that will generate periodic orbits, but for obtuse triangles there are no known constructions.Using computer assisted searches, R.E Schwartz has shown that periodic orbits exist for obtuse triangles when the largest angle is less than 100°.

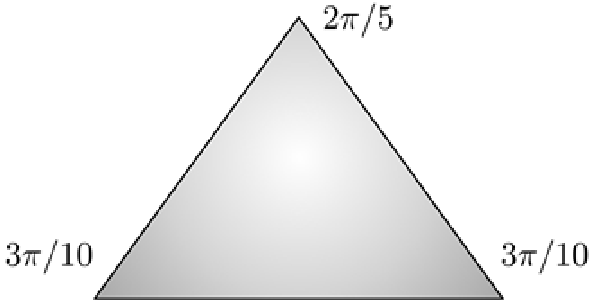
There are other classes of polygons which are known to have periodic orbits. In inner billiards a polygon is called *rational* if every vertex angle is a rational multiple of π. In this case every orbit that departs normal to a side is either periodic or terminates in a corner. This implies that there are a continuum of periodic orbits for such polygons. This result has been extended to show that for regular polygons, every orbit that does not terminate is either periodic or uniformly distributed (W.A.Veech). Polygons with this property are called 'ergotically optimal'. There is a slightly weaker condition where 'uniformly distributed' is replaced by 'dense'. These polygons are sometimes called 'topologically optimal'. Clearly ergotically optimal implies topologically optimal. (For billiards on the circle Jacobi's theorem implies that the orbit is dense when the rotation number k is π-irrational, but this does not imply that the iterates are uniformly distributed. However a theorem of Wely and Kronecker shows that the orbit is uniformly distributed, so in this sense the circle is ergotically optimal.)

There are billiards tables which have orbits which are dense but not uniformly distributed. Veech, Masur and Smillie have studied an example consisting of a 2 by 1 rectangle with a barrier as shown on the left below. When the length of this blue barrier is rational, the table is ergotically optimal, but when this length is irrational, there are orbits which are neither dense nor closed and there are dense orbits which are not uniformly distributed. These latter orbits are 'robust' and have Hausdorff dimension ½. The L-shaped table on the right shows an orbit which is neither dense nor closed.





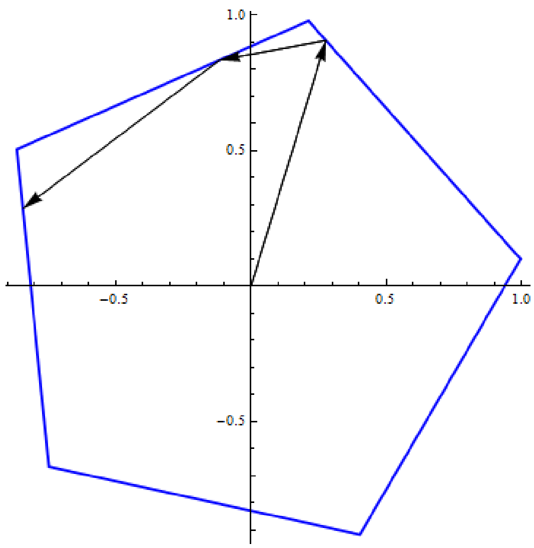
A recent result by Cheung,Hubert and Maser shows that the triangle belowis topologically optimal and not ergotically optimal. The values given are the interior angles. Note that this triangle ‘tiles’ the regular pentagon.



Theorem . *The billiard dynamics on the isosceles triangle with angles*$(2\pi/5, 3\pi/10, 3\pi/10)$ *are topologically optimal but not ergodically optimal: for each direction, either all infinite trajectories are closed or all are dense, but there exist trajectories which are dense but not uniformly distributed.*

Recently there have been results showing that every triangle which is sufficiently close to an isosceles triangle has a periodic billiard orbit.

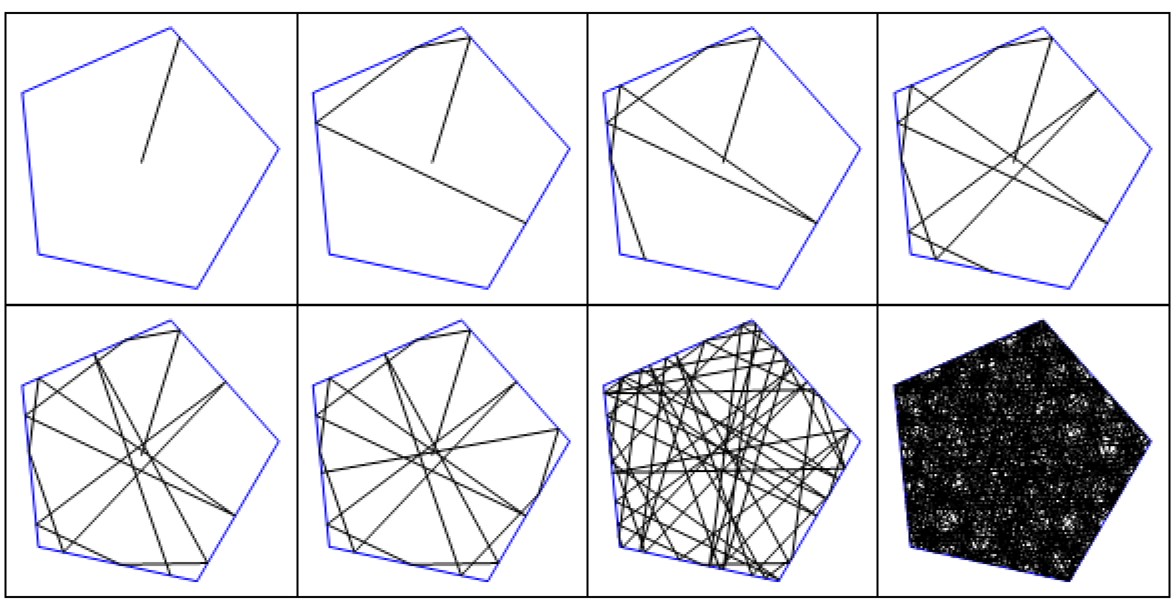
Below is an example of a billiard trajectory inside a regular pentagon.Any regular polygon is trivially ‘rational’ so there is no shortage of periodic orbits, but they are still ‘special cases’. Most orbits are uniformly dense. Below is an example using an initial trajectory vector centered at the origin which makes an angle of 73° with the x axis.



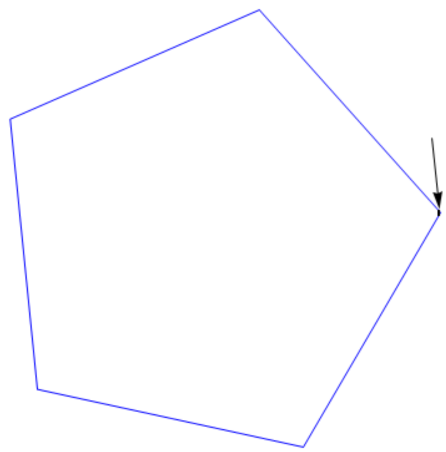
Orbit[n_]:=BilliardPolygon[5,{0,0},73Degree,n];

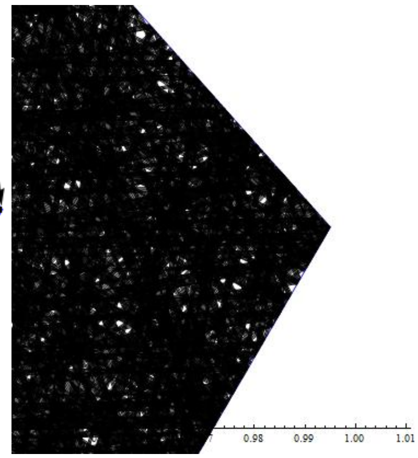
**Orb[n_]:=Graphics[{{Thickness[0.005],Blue,Line[{Cos[#],Sin[#]}&/@Range[0.1,2 \[Pi]+0.1,2 \[Pi]/5]]},
Line[Orbit[n]] },AspectRatio->Automatic];**

GraphicsGrid[{{Orb[0],Orb[3],Orb[6],Orb[9]},{Orb[12],Orb[15],Orb[50],Orb[300]}},Frame->All]



On the scale above, the pentagon will appear black after just a few thousand iterations. Below is an enlargement of the small region shown by the arrow after 20,000 iterations



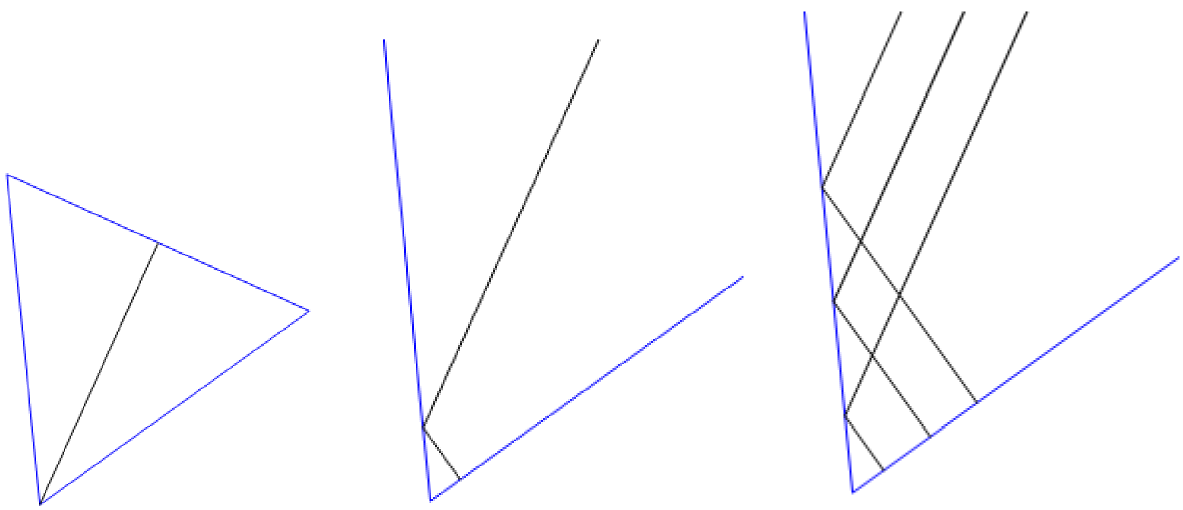


The software in InnerBilliards.nb is based on the notebook by Eric W. Weisstein (2007) at http://mathworld.wolfram.com/notebooks/PlaneGeometry/Billiards.nb. He attributes the polygon portion to Robert Dickau. This code has not been updated and it only deals with regular polygons. Below is an example with a regular triangle and an initial trajectory which is nearly orthogonal.

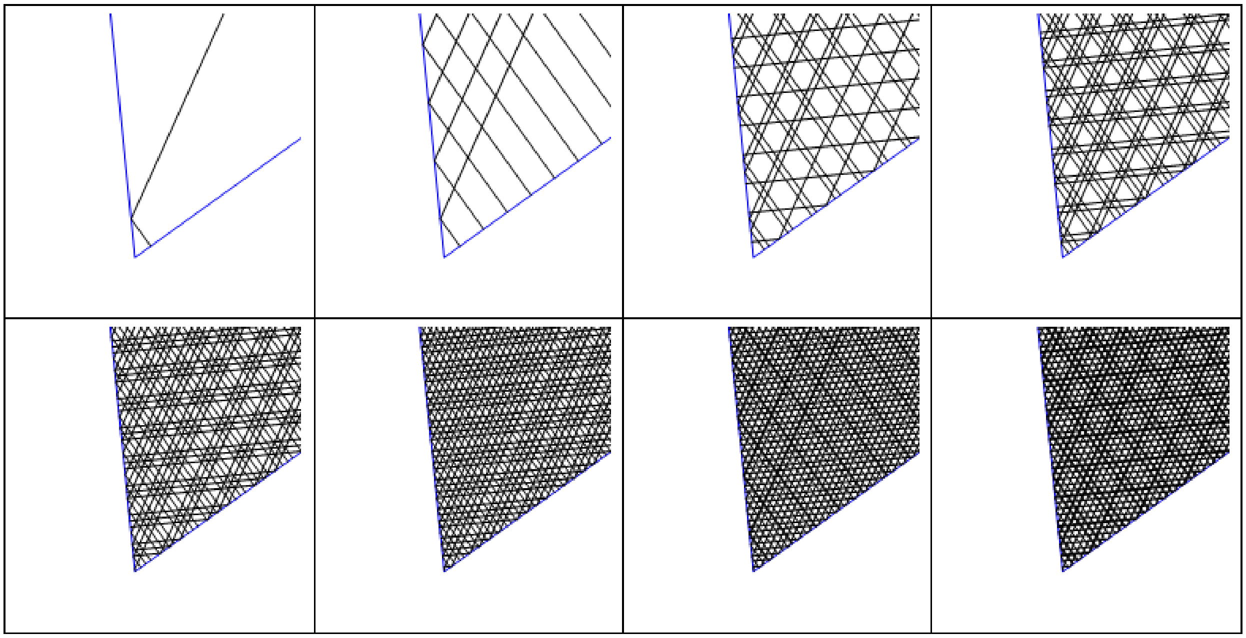
Example**: Orbit[n_]:=BilliardPolygon[3,{0,0},65.75Degree,n];**

Orb[n_]:=Graphics[{{Thickness[0.005],Blue,Line[{Cos[#],Sin[#]}&/@Range[0.1,2 \[Pi]+0.1,2 \[Pi]/3]]},Line[Orbit[n]] }, AspectRatio->Automatic,PlotRange->{{left,right},{bottom,top}}];

Orb[2] is shown on the left and the lower corner is enlarged in the middle plot. The return orbit is almost periodic. On the right is the corner after 10 iterations.



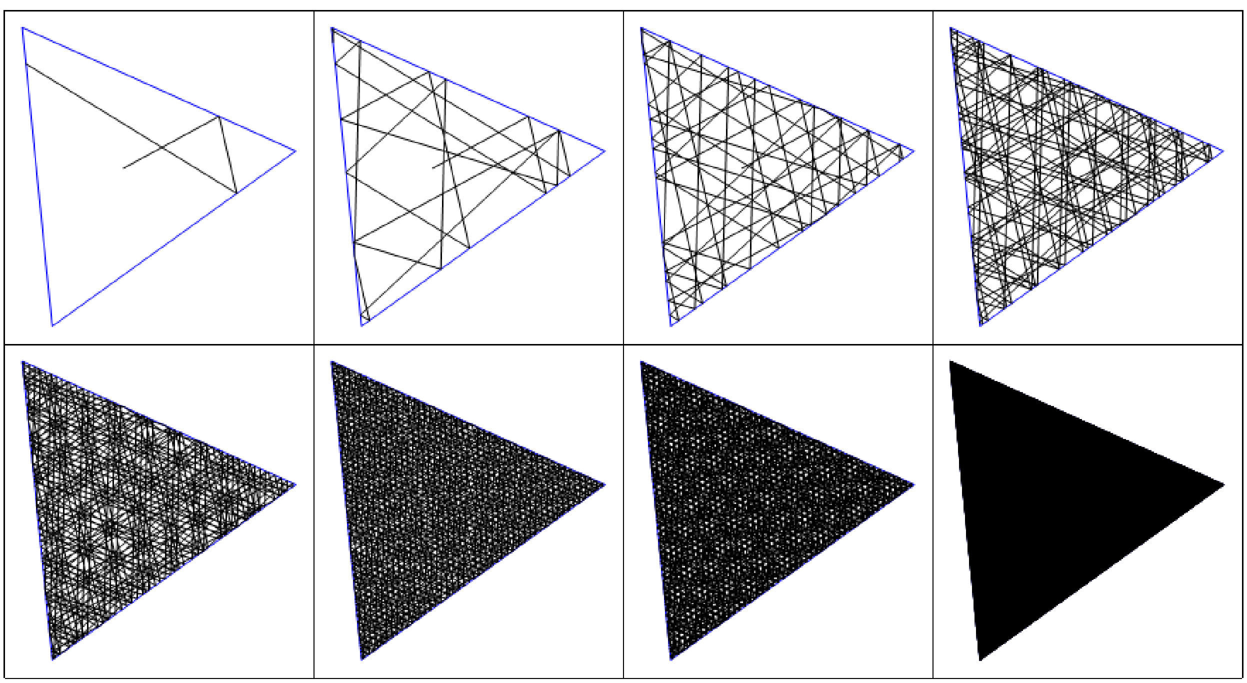
GraphicsGrid[{{Orb[2],Orb[50],Orb[10000],Orb[20000]},{Orb[30000],Orb[40000],Orb[50000],Orb[60000]}},Frame->All]



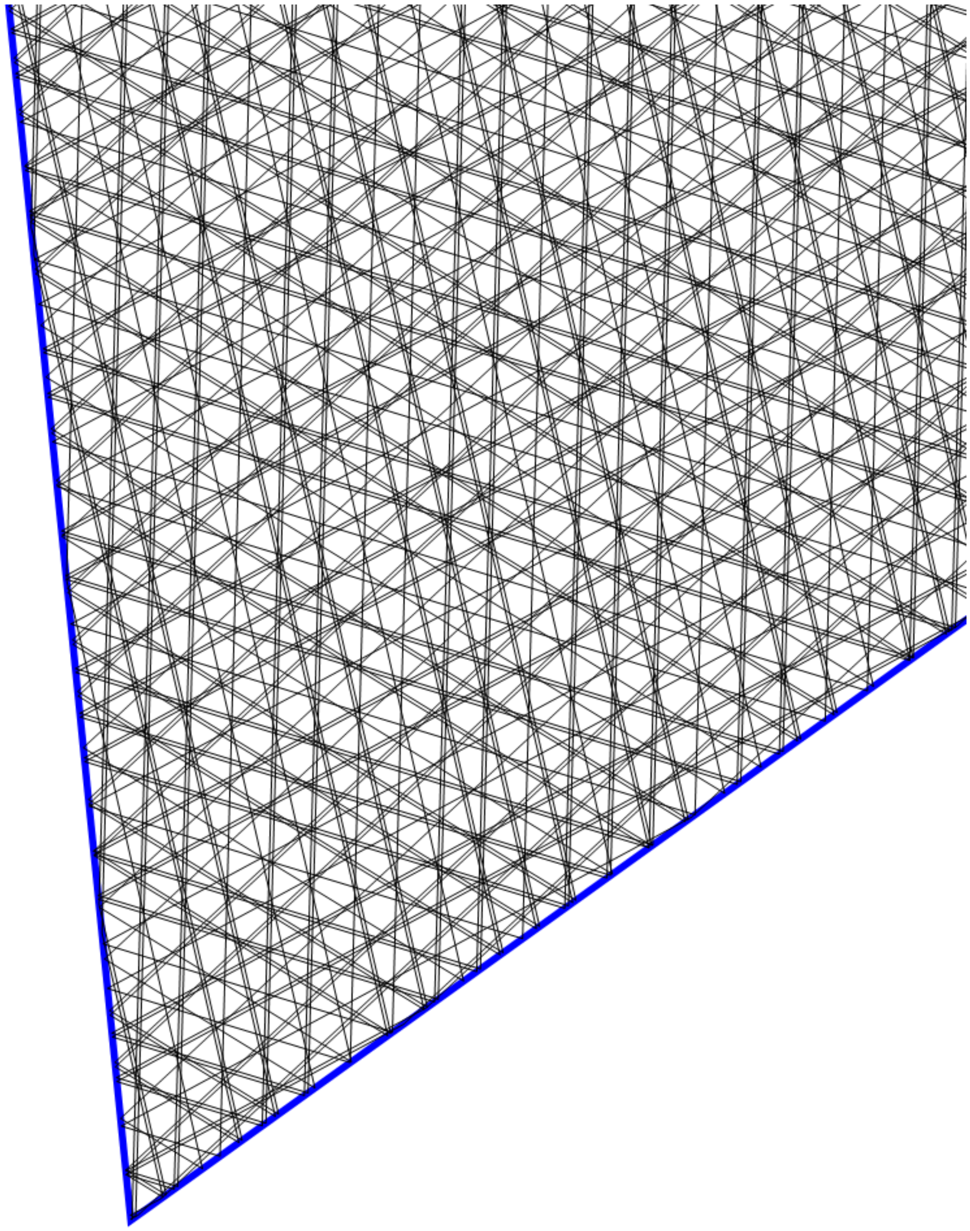
Below is a more ‘typical’ example:

Example**: Orbit[n_]:=BilliardPolygon[3,{0,0},28.45732Degree,n];**

GraphicsGrid[{{Orb[2],Orb[20],Orb[50],Orb[100]},{Orb[200],Orb[300],Orb[500],Orb[5000]}},Frame->All]



Below is an enlargement of the lower corner after 50,000 iterations.



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