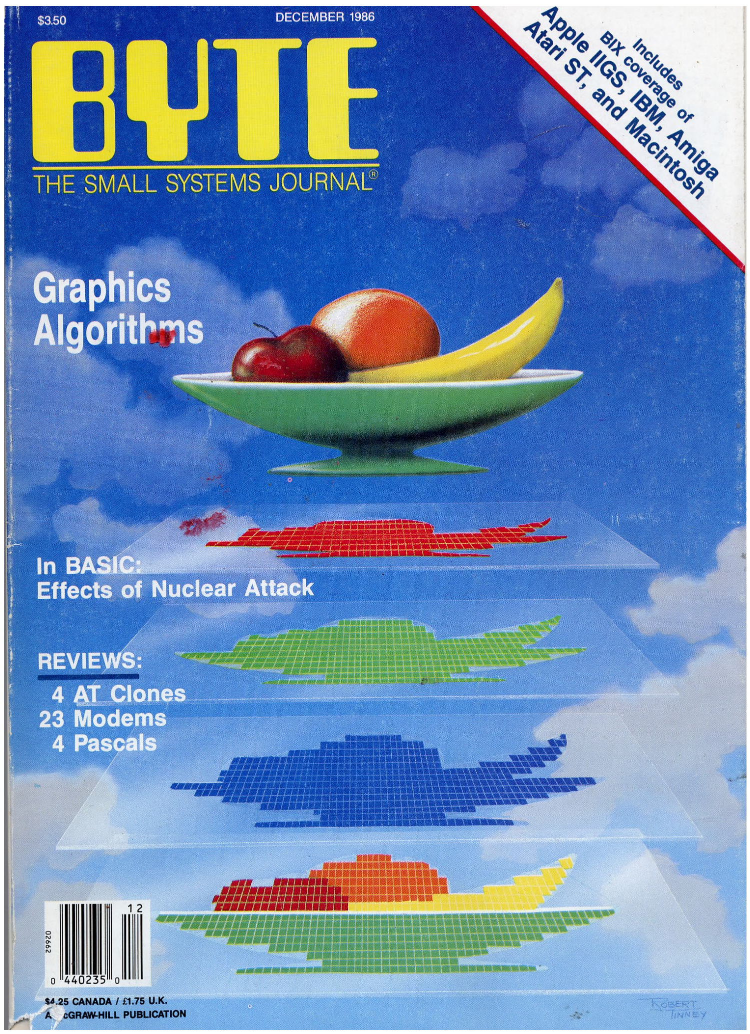
[H1] Hughes G.H. Henon Mapping with Pascal, BYTE – The Small System Journal, Vol 11, No. 13, Dec 1986

In the 1980's computers were just becoming affordable and in 1981 IBM released the IBMPC at about $1500. It was based on the Intel 8088 CPU and used 'open' architecture so vendors could provide peripherals and software. This BYTE magazine article was 'hands-on' so anyone with a Pascal compiler could reproduce the graphics. BYTE was at this time the premier PC magazine in the world and they paid $500 for the article which was thoroughly reviewed. This article was also accepted by Scientific American, but it was too late to change publishers, so BYTE gave them permission to publish excerpts which appeared in July, 1987 as shown below.



HENON MAPPING WITH PASCAL

BY GORDON HUGHES

A window on the world's endless complexity

IN 1968, MICHEL HENON of the Institute for Astrophysics in Paris proposed a simple quadratic mapping of the plane as a model for the study of dynamical systems such as the motion of asteroids, satellites, or charged particles in an accelerator. Henon's mapping (see the text box "Creating a Henon Mapping" on page 170) is based on George Birkhoff's discovery in 1917 that you can reduce the study of conservative systems with two degrees of freedom to the study of area-preserving mappings of the plane. Thus, Henon set out to find an area-preserving mapping that was simple in nature but retained all the characteristics of more complicated mappings.

Although the mapping Henon proposed is easy to describe and program, it yields results of great complexity. Since Henon mappings simulate the behavior of physical systems, they indicate that many such systems are more complex than previously imagined. Mathematicians and physicists are only beginning to understand the nature of this complexity and what it says about physical systems such as the asteroid belt. In a series of results during the years 1954 to 1963, mathematicians A. N. Kolmogorov, V. I. Arnold, and J. Moser provided a partial explanation for the strange behavior of such systems. These results are now known as the KAM theorem. It is an important theorem in modern physics and has aroused a great deal of interest.

WHAT IS THE KAM THEOREM?

The KAM theorem explains mathematically what happens when a small external force disturbs a stable dynamical system, such as a satellite in orbit around Earth. One such disturbance that satellites regularly undergo is the uneven pull of gravity due to Earth's bulge at the equator. For the planets and the asteroids orbiting the sun, the chief disruptive force is the pull of Jupiter. It causes a perturbation in their orbits so that the orbits are not precisely elliptical. The question that the KAM theorem addresses is whether these slight irregularities have long-term effects leading to eventual instability.

The theorem shows that under small disturbances a stable system undergoes changes but remains stable except for microscopically small bands of potential instability corresponding to "resonances" between the original system and the disturbance. With an asteroid disturbed by Jupiter, such a resonance will occur if the ratio of their periods is a rational number: For example, a 2/5 resonance occurs if two orbits by Jupiter take the same amount of time as five orbits of the asteroid. The KAM theorem proves that as long as the disturbances remain small the relative size of these resonance bands is insignificant and stability is assured.

Figure 1 is a Henon mapping that simulates a system undergoing successively larger disturbances. The inner curves represent a system's reaction to small disturbances and show that the system is altered slightly but remains stable. (The microscopic resonance bands are much too small to be visible in this scale.)

However, if the disturbance in-

(continued)

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creases in magnitude past a certain threshold value, some of the resonance bands will suddenly widen. In figure 1, for example, the first noticeable widening is the resonance band with six "islands," indicating a 1/6 resonance. An asteroid with a period 1/6 that of Jupiter would find itself on such a resonance band.

With even higher disturbances, the resonance bands might dominate the system's behavior, as indicated by the outer band of seven large islands. The scattered dots around these islands indicate areas of instability. Similar areas exist between the islands of each resonance band. An asteroid caught in one of these regions could experience erratic behavior and even be thrown from its orbit, as indicated by the faint dots escaping around figure 1. Such resonances are believed to cause the Kirkwood gaps in the asteroid belt.

Scientists have known for some time that resonances between two interacting forces can lead to instability in the form of erratic or extreme behavior. In electrical circuits, such resonances are sometimes exploited to amplify selected signals. But the KAM theorem shows that, at least for small interactions, the resonances don't lead to abrupt changes in behavior. The system "stretches" smoothly and doesn't break.

EXTENDING THE KAM THEOREM

Although the KAM theorem is a landmark result in the study of stability, it is concerned entirely with very small disturbances. Thus, it doesn't address the question of the sudden widening of the bands, nor does it attempt to explain the exact nature of the resonance bands. The KAM authors knew that these regions would be very complex in nature due to the buildup of resonances. The close-up in figure 2 gives some hint of the complexity involved. The pointed region at the center is an unstable hyperbolic fixed point that always exists between stable islands. You can see two thin resonance bands just to the left of this point. At the top and bottom are the edges of the large islands. The smaller islands are secondary resonances of the large islands. These in turn generate third-order resonances and the process continues forever with endless chains of smaller and smaller islands extending into the chaotic center region. The central region appears to be a scattering of random points but is actually a complex interweaving of tiny islands and their corresponding hyperbolic points.

As far back as 1892, Henri Poincaré predicted the possibility of chaotic behavior in certain celestial motions due to these resonances. Since then, a number of prominent physicists and mathematicians have wrestled with the question. George Birkhoff invented an elaborate code called "symbolic dynamics" to represent the structure of these regions. Recent studies by mathematicians such as Stephen Smale center on using mathematical topology, an outgrowth of geometry.

These questions are not just theoretical. The KAM theorem applies to a wide range of ordinary physical systems. Therefore, the study of resonance thresholds and the resonance bands has a number of practical ramifications for items in the physical world, from atomic particles to galactic clusters. At the 1985 NATO conference on the stability of the solar system (see Szebehely in the bibliography), a number of results centered on the KAM theorem and its consequences for the stability of the planets and satellites. Because stability is concerned with a system's long-term behavior, many years may pass before scientists observe any effects. The solar system's long-range stability is still an open question. For asteroids, satellites, and charged particles in an

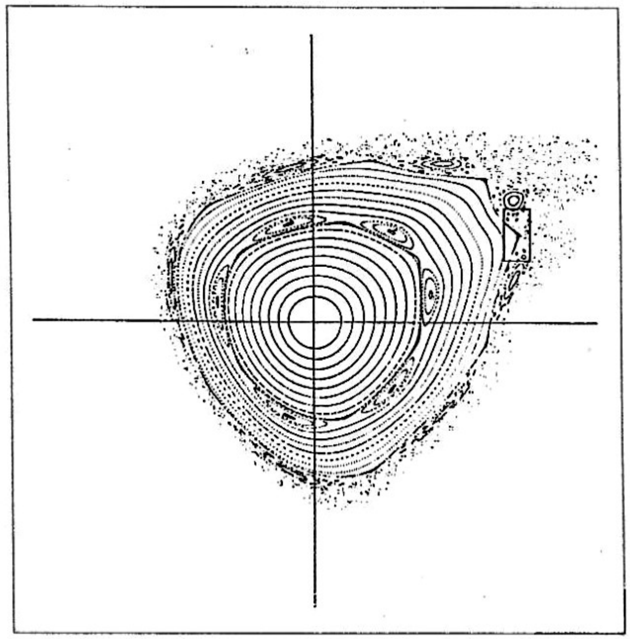

Figure 1: A *Henon mapping with phase angle A = 1.111 radians. The scale is −1.2 to 1.2 on both the x and y axes. Thirty-eight orbits are shown, each consisting of 700 iterations of a single starting point. The outlined area is enlarged in figure 2. (Use the parameter file* A111P.01 *with* HENON2.COM *or the values above with* HENON1.COM *to create this mapping.)*

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The Hénon mapping generates different figures for **a** = *.264* **(left)** *and* **a** = *1.5732* **(right)**

reasons the core of CHAOS2 does not use standard initial values for its iteration variables. They must be input by the computer programmer.

CHAOS2 is complete when its core is preceded by an input statement that allows the programmer to select the value of a. As in CHAOS1, each new value of a leads to a new system. But because the system is two-dimensional, a sampling of orbital plots takes up all available room; one cannot systematically vary the control parameter a without invoking chaos of an unwanted kind.

The user of CHAOS2 therefore specifies an initial orbit by typing in the coordinates of a point on it. Sitting back, he or she watches in fascination as the orbit is plotted. It might turn out to be a curve (traced not continuously but intermittently) or it might turn out to be something a little stranger. For example, the bottom illustration on the opposite page displays a succession of 38 orbits in a Hénon mapping with the value of a set at 1.111. From the center of the plot outward the orbits form a nest of closed curves until the sudden appearance of small "islands": individual orbits wedged between the larger nested ones. Farther out the nested orbits continue until the onset of chaos. In the outer reaches of the phase plot more islands appear, along with a random sprinkling of points that denotes the onset of chaos. One of the chaotic areas (outlined by a rectangle in the illustration) is shown in magnified form. Readers who want to magnify Hénon diagrams are warned to use the most precise arithmetic available on their machines.

As I have mentioned, Hénon mappings represent a great variety of conservative systems, such as asteroids orbiting the sun. Unfortunately the orbits in the diagrams are not the orbits of the asteroids but phase plots of those orbits. In the diagram just described the horizontal axis may represent the position of an asteroid in terms of its distance from the sun. The vertical axis may represent the radial velocity, or the rate of change, in this distance. Each point on the orbit computed by the Hénon mapping represents the radial distance and velocity of an asteroid at a specific angular position with respect to the sun, that is, when the asteroid passes through a vertical plane making that angle with the sun. Successive points computed by the mapping represent the asteroid's successive reappearances on the plane. The islands mentioned above are resonance bands due to perturbations in the asteroid's orbit by larger bodies in the solar system such as Jupiter. In the chaotic regions the radial position and velocity of an asteroid will vary in an essentially random way every time an asteroid revisits the specified plane. Its motion is unpredictable. Almost anything can happen.

On the aesthetic side it is worth looking at some other plots generated by Hénon mappings; quite apart from physical interpretations, there is a whimsical quality present, as is seen in the illustration above. They resemble strange, aquatic creatures.

Readers wanting to learn more about Hénon mappings should get a copy of the December 1986 issue of *Byte* magazine. There Gordon Hughes, a professor of mathematics at California State University, has engagingly described some of the relevant physics and mathematics underlying Hénon mappings. PASCAL programs are also listed.

A reader in Holland, Peter de Jong of Leiden, has already suggested some other iteration formulas that produce bizarre shapes and images. He recommends the four-parameter iterations $x \leftarrow \sin(ay) - \cos(bx)$ and $y \leftarrow \sin(cx) - \cos(dy)$. Begin with x and y both set equal to 0. Then, to get the figure de Jong calls "chicken legs," try $a = 2.01$, $b = -2.53$, $c = 1.61$ and $d = -.33$. The respective values -2.7, $-.09$, $-.86$ and -2.2 yield a "dot launcher," and the values -2.24, $.43$, $-.65$ and -2.43 produce a "self-decorating Easter egg."

Readers are free, like de Jong, to invent their own iteration formulas and to experiment with them. Anyone finding particularly attractive (or puzzling) chaos is hereby invited to send it to me in care of this magazine. Crutchfield has kindly agreed to correspond directly with those readers whose puzzlement I am not likely to satisfy. He can be reached at the Department of Physics, University of California, Berkeley, Calif. 94720.

Readers with one-track minds have by now undoubtedly solved last month's problem involving the interchange of two cars across a weak bridge. The cars are on a circular track, and an engine occupies another track connected to the circular one by a switch. The bridge is strong enough to hold one car but not the engine. How can the engine switch the cars?

The engine enters the circular track, goes to car A and pushes it onto the bridge. Then the engine backs around the track to car B, couples with it, pushes it to the edge of the bridge and couples B with A. Chugging back to the switch, the engine and two cars back onto the straight track, where A is uncoupled. Next the engine takes B back to the bridge, leaving it uncoupled there. Finally the engine circles the track, pulls B off the bridge to its new position and then retrieves A.

In the April column on computer music I left readers to ponder how to obtain long, nonrepeating sequences of notes by selecting numbers modulo m. The method of selection involved starting with a seed number and from then on continually multiplying by a number a, adding another number b and reducing the result by taking the remainder on division by m. If the numbers a and m are relatively prime (have no common factor larger than 1), the sequence will be the longest one possible. It will also produce the strangest music.

Peter de Jong notes that he has created strange music through chaos. Readers can create similar sounds by converting the numbers generated by CHAOS1 into musical notes. Outside chaotic zones there will be simple, repetitive musical phrases; inside the zones will be the very sounds of chaos.