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# Webs and periods of orbits

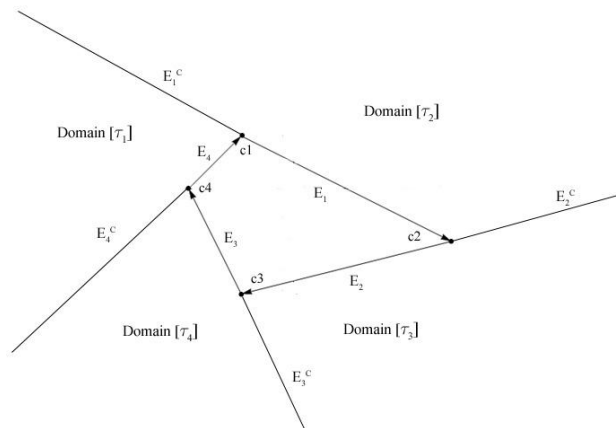
Any family of piecewise continuous functions will have points where the maps are discontinuous. Usually it is assumed that the maps are undefined at these points. The set of such points is sometimes called the 'exceptional set' or the 'separatrices'. In the complement of the exceptional set, neighboring points tend to have similar dynamics. We will define the exceptional set for the Tangent Map below. It will also be known as the 'web'.

For an n-gon M, the Tangent Map  $\tau$  decomposes into n maps:

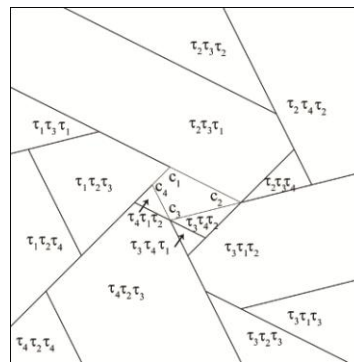
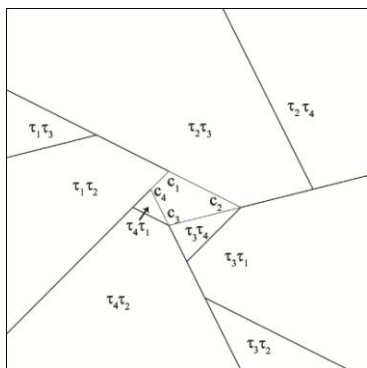
$$\tau_k(p) = -p + 2c_k \text{ where } c_k = M[[k]].$$

Then the Tangent Map is defined as:  $\tau(p) = \tau_k(p)$  iff  $c_k$  is the 'supporting' vertex for p.

**Example:** The lattice polygon:  $M = \{-1, 1\}, \{3, -1\}, \{-1, -2\}, \{-2, 0\}$  is shown below with  $M[[1]] = c_1 = \{-1, 1\}$ . The edges are numbered clockwise. The Tangent Map reduces to four maps. The 'trailing' edges  $E_1^c, E_2^c, E_3^c, E_4^c$  form the level 0 exceptional set W. If we include the edges of M, then W is connected. This is called the level 0 web,  $W[0]$ . These are the only points external to M, where  $\tau$  is not defined. The complement of the web consists of disjoint open regions which we call the level 1 domains. The four level 1 domains for M are shown below.



Since  $\tau$  is not defined on  $W[0]$ ,  $\tau^2$  is not defined on the inverse image of  $W[0]$  so  $\text{Dom}(\tau^2) = \text{Dom}(\tau) - \sigma(W[0])$  where  $\sigma = \tau^{-1}$ . The union of  $W[0]$  and  $\sigma(W[0])$  is called the level 1 web as shown on the left below. The level k web is  $W[k] = \text{NestList}[\sigma, W[0], k]$ . This is a 'working definition', formal definitions will follow.

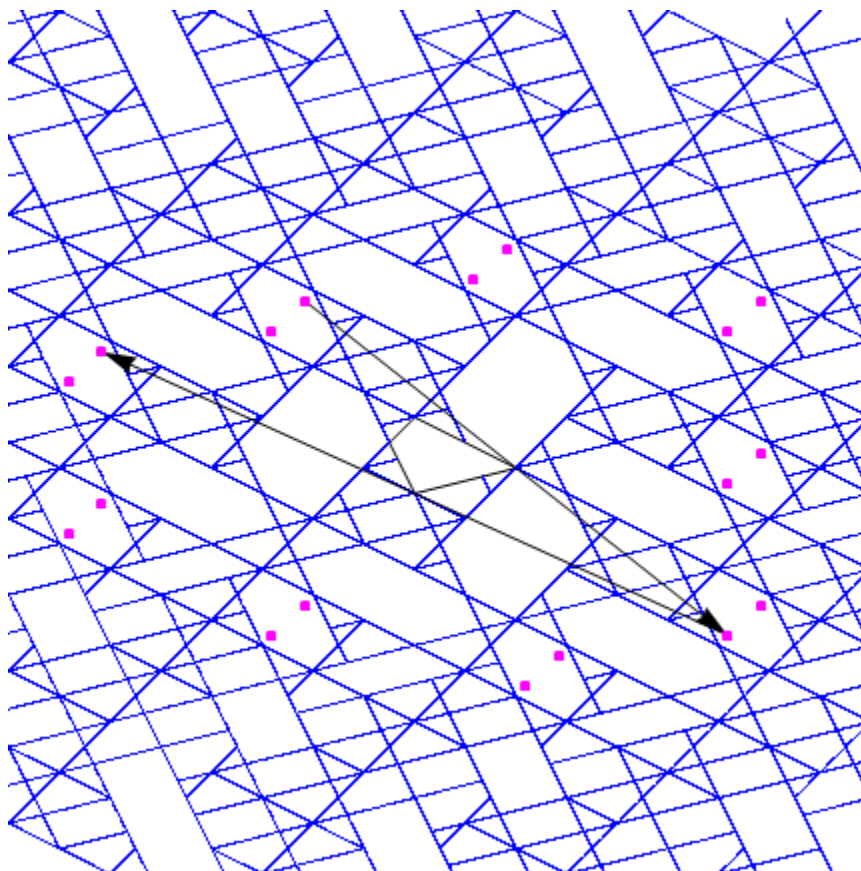


Each region in the complement of the web can be labeled with the corresponding 'corner sequence' and these in turn can be used to classify the dynamics. Any level  $k$  subdomain will map undivided for at least  $k$  iterations of  $\tau$ , so we can define periods of subdomains in the same way that we define periods of points.

Below is a portion of the level 300 web for  $M$ , showing 9 congruent subdomains which map to each other in a periodic fashion. Since  $\tau$  is an inversion, the points return to the original region after 9 inversions. Because this is odd, these points will have to make two circuits, and their period will be 18. But there is a central point which feels no inversions, so the center has period 9. This scenario is called 'period doubling' and it is generic for all regions which have odd period.

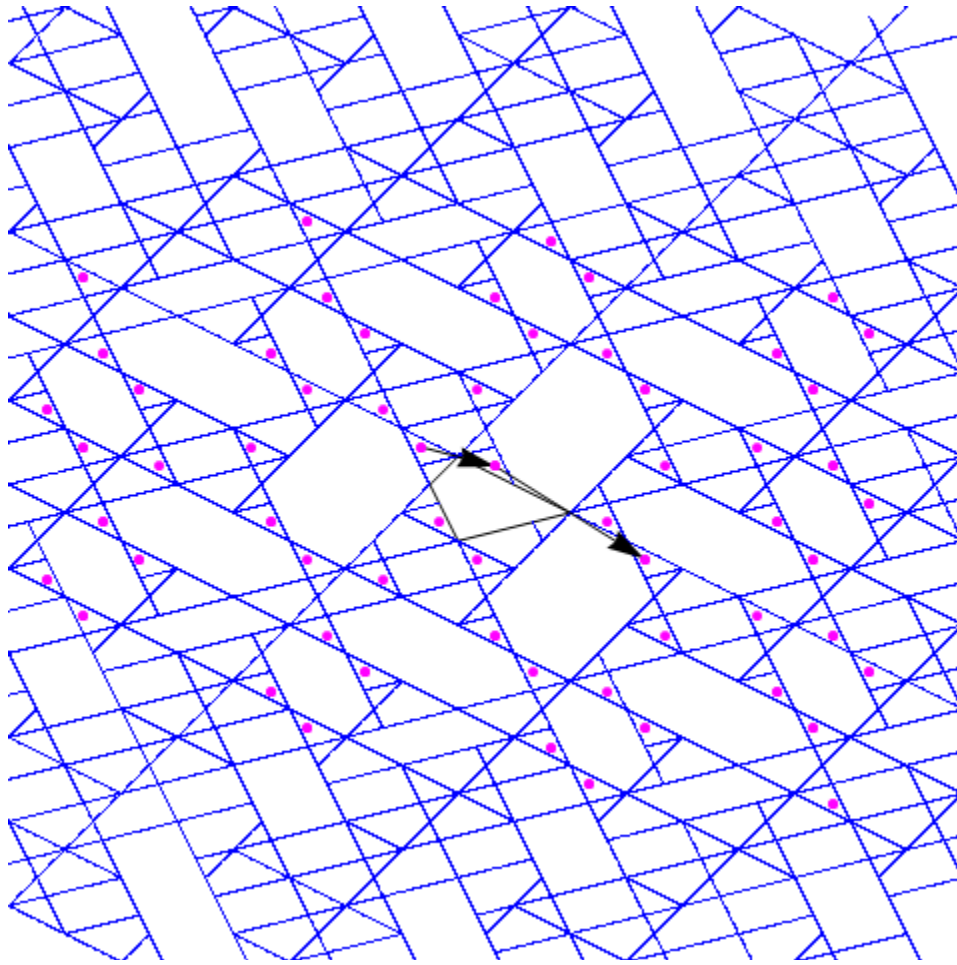
Our convention is to call this a period 9 ring. By the same convention, the ring of three large 'Dads' below, would be called a period 3 ring, even though all points are period 6, except for the center.

$q1 = \{-5.333, 5.583\}$ ;  $K = V[q1, 200]$ ;  $\text{Period}[K]$  Out = 18.



The corner sequence for this orbit is  $\text{IND}[q1, 18] = \{3, 4, 2, 4, 2, 3, 1, 3, 4, 3, 4, 2, 4, 2, 3, 1, 3, 4\}$ . Note that it is period 9 with step sequence  $Vp[q1, 9] = \{1, 2, 2, 2, 1, 2, 2, 1, 3\}$ .

If a subdomain has even period then any point will return after an even number of inversions. Since the entire subdomain must map to itself, this mapping must be the identity. Therefore every point has the same even period. For example using  $M$  from above  $q_1 = \{-2.345, 1.353\}$  has period 60 and every other point in this region has the same period, so there are 60 congruent regions as shown below.



The corner sequence and step sequence are period 60. Below is the step sequence:  $V_p[q_1, 60]$   
 $\{1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 1, 3, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, 2, 2, 2, 2, 2, 1, 2, 2, 1, 3, 1, 2, 2, 2, 1, 2, 2, 2, 1, 1, 3, 1, 2, 1, 2, 1, 2, 1, 1, 1, 1\}$

**Lemma:** If a subdomain has even period  $k$ , then all points in the subdomain have period  $k$ . If the subdomain has odd period  $k$ , the all points in the subdomain have period  $2k$  except for the center which has period  $k$ .

## The inverse web and the forward web

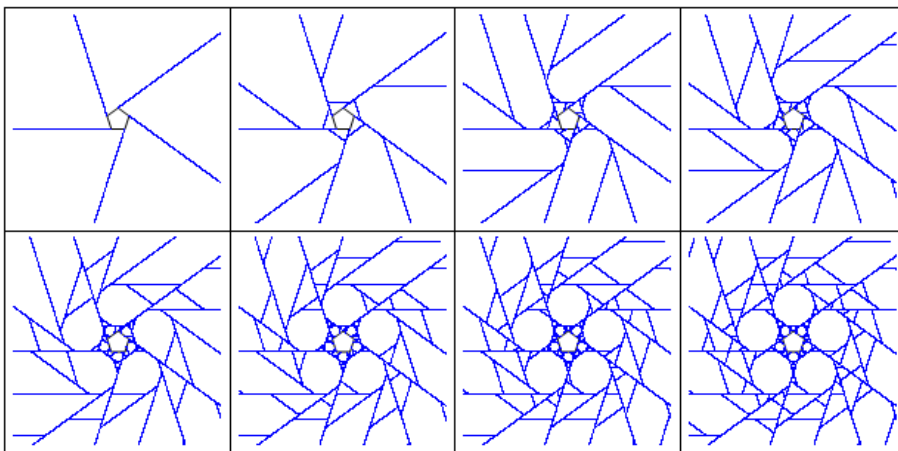
The web  $W$  was obtained using the trailing edges and the inverse map  $\sigma$  so we will call this the 'inverse web'. To find periodic points, we should also exclude the forward edges because no point on a forward edge can have an inverse image and this implies that no point on a forward edge can be periodic. This applies equally to the (forward) images of forward edges. We will call this the 'forward web'.

From a practical standpoint it is easier to generate the forward web because it uses the original tangent map  $\tau$ . All the notebook packages generate forward webs. To generate a true inverse web, we would need to implement the inverse map  $\sigma$ , which is the same as  $\tau$  but with a counter-clockwise orientation. For regular  $N$ -gons this counter-clockwise version of the polygon is identical to the original so we can generate a forward web and imagine that we are working with a polygon that originally had counter-clockwise orientation.

All the notebooks share the same web algorithm: `web[stepsize_,length_,depth_,offset_]` which scans the forward edges at a given step size out to the given length and generates  $k$  images of each point where  $k = \text{depth}$ . The routine called `WebPoints` collects these edge scans together into a file for plotting.

Example: The regular pentagon  $N = 5$

```
Gr[depth_] := Show[Graphics[{AbsolutePointSize[1.0], poly[Mom], Blue,
  Point[WebPoints[.01, 12, depth]]}, PlotRange -> {{left, right}, {bottom, top}}]];
GraphicsGrid[{{Gr[0],Gr[1],Gr[2],Gr[3]},{Gr[4],Gr[5],Gr[6],Gr[7]}},Frame->All]
```

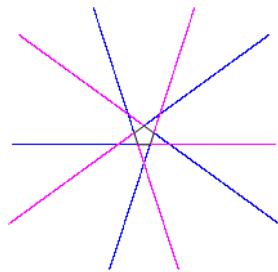


With some practice it is possible to track the evolution of the webs by swapping domains and ranges. This is called cobwebbing. For a function of one variable, the line  $y = x$  performs this magic. For  $N = 5$ , by level-3 the inner edges of Dad have sealed off the star region because the 3rd iteration aligns with a trailing edge of Mom. All prime  $N$ -gons have invariant star regions after  $N-2$  iterations.

There are times when it is useful to generate the true inverse web of a polygon. For example to associate regions with their corner sequences, the forward web will not work, because only the inverse web can distinguish subdomains properly at each iteration. Of course once regions are periodic the two webs yield the same information. In the interim it may be useful to generate both webs and take their union. We will do this with non-regular polygons below.

Example: Using the regular pentagon. **W0 = WebPoints[.01, 12, 0]**. This will generate the depth 0 (Forward) web, which consists of the forward edges (as shown in blue below). To generate the backwards edges, which make up the depth 0 Inverse web, use **Rt[p\_]:=ReflectionTransform[{1,0}][p]**; This will reflect points about the normal to {1,0} which is the y-axis, so **Rt[W0]** will yield the trailing edges as shown below in red.

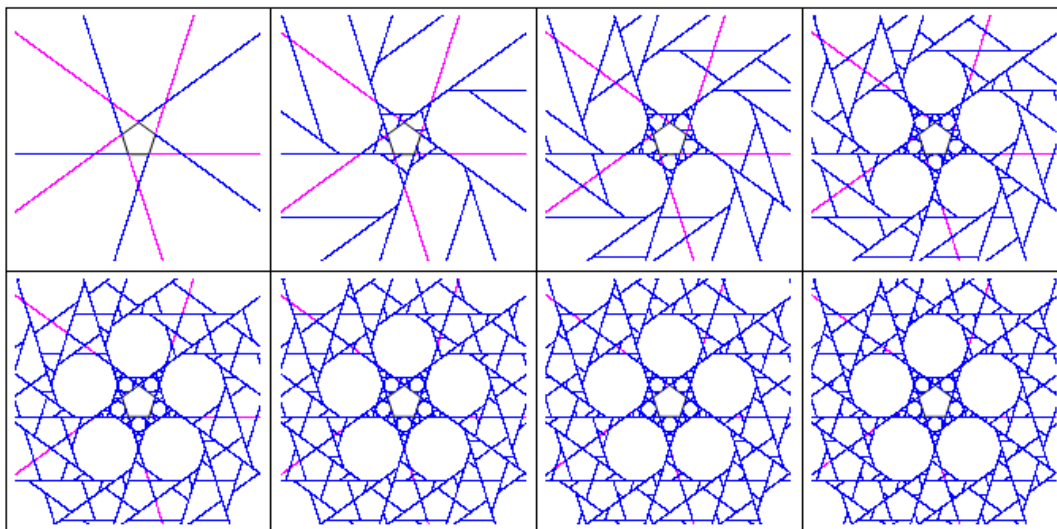
```
Graphics[{poly[Mom],AbsolutePointSize[1.0], Blue, Point[W1], Red, Point[Rt[W0]]}]
```



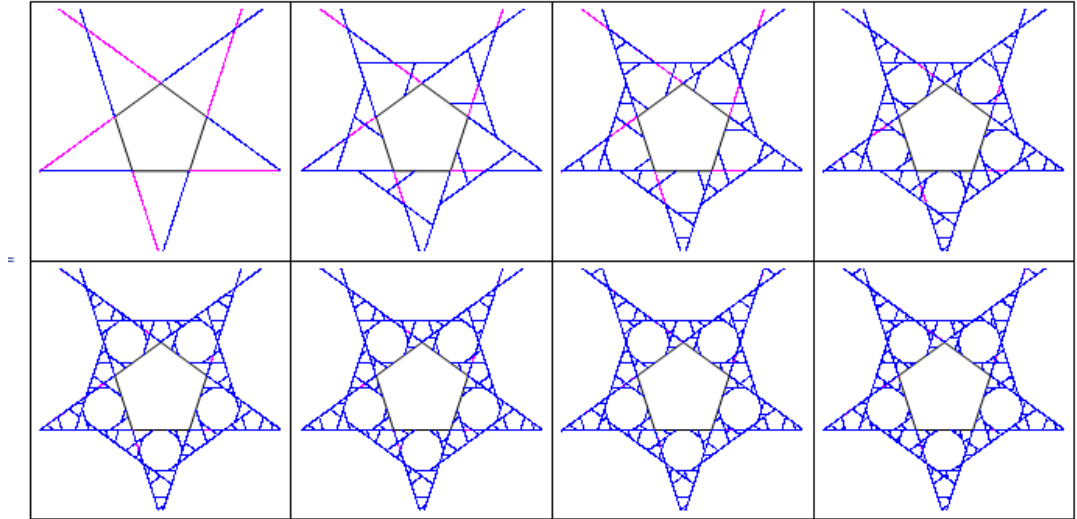
The blue edges can only map to the red and conversely. In the plots below we will map the blue (Forward) edges and keep the red edges fixed.

```
Gr[depth_]:=Show[Graphics[{AbsolutePointSize[1.0],poly[Mom],Magenta,Point[Rt[W]],Blue,Point[WebPoints[.01,12,depth]]},PlotRange->{{left,right},{bottom,top}}];
```

```
GraphicsGrid[{{Gr[0],Gr[3],Gr[6],Gr[9]},{Gr[12],Gr[15],Gr[18],Gr[21]}},Frame->All]
```



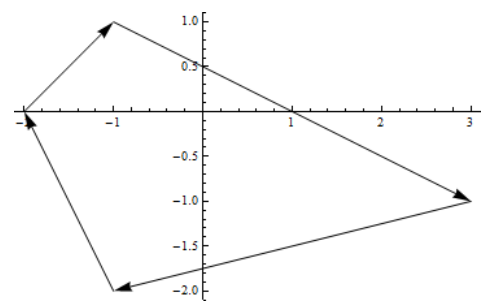
By level 21 most of the forward edges have mapped to trailing edges, but there will always be 'hot' points because the web is always evolving - even in the central star region. Below is a close up of this evolution. It is no surprise that the 'hot' spots are converging to the non-periodic limit points.



From a practical standpoint, the forward web of a regular polygon will give correct dynamics after just a few iterations. The webs are just visual aids and none of the code depends on the web. The forward webs are not 'incorrect' but the user should understand what information they contain, and that information is the dynamics of the inverse map  $\tau^{-1}$ . For periodic regions this is the same information as  $\tau$ , but for non-periodic regions there will always be differences between the forward and trailing webs. We will show some examples with the Penrose kite.

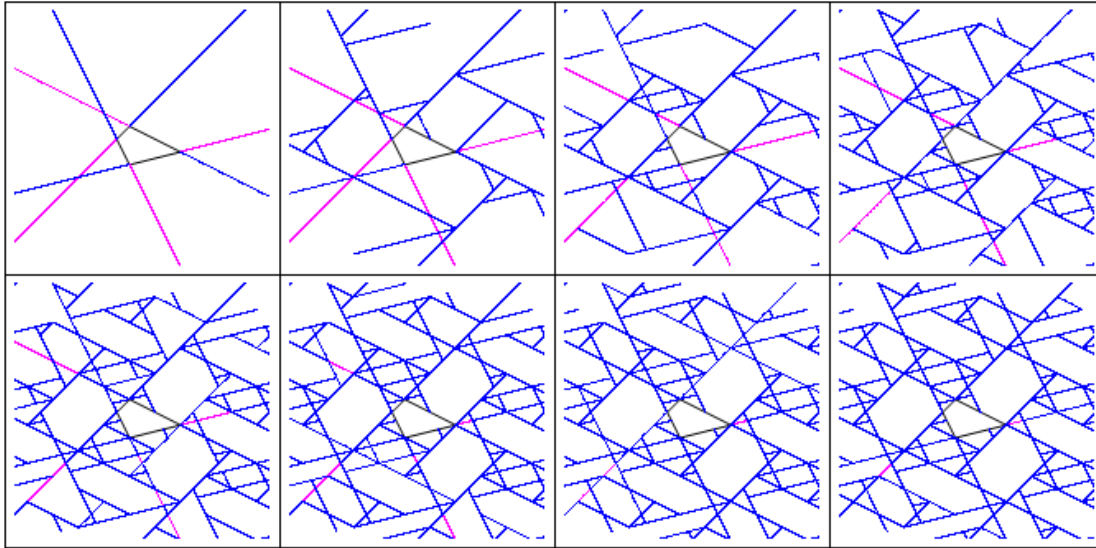
**Webs of non-regular polygons**

To see what happens for a typical non-regular polygon, we will use the lattice polygon defined above:  $M = \{-1, 1\}, \{3, -1\}, \{-1, -2\}, \{-2, 0\}$

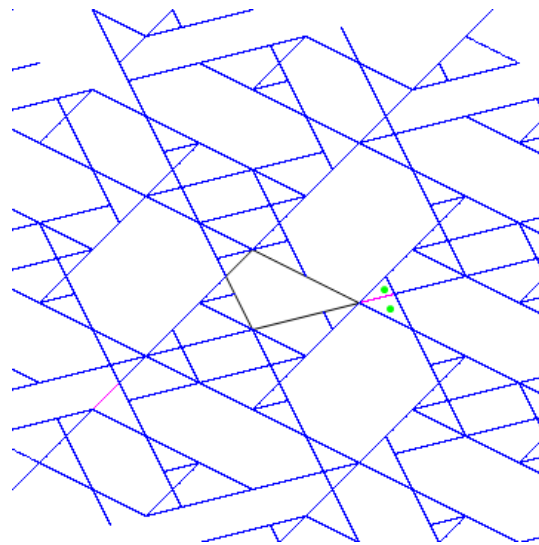


The reflection transformation  $R_t$  from above works for any polygon and we have applied it here to obtain the trailing edges shown below in red. We will once again map the blue forward edges using every third iteration:

```
GraphicsGrid[{{Gr[0],Gr[3],Gr[6],Gr[9]},{Gr[12],Gr[15],Gr[18],Gr[21]}},Frame->All]
```



Below is an enlargement of the level 21 web from above. It should be clear that the two green points have different orbits from the very first iteration, but after 21 iterations it still looks like they are related. It is not until the level 51 web that this region is partitioned correctly. The trailing edges partitioned it correctly on the first iteration. The top point has period 154 and the bottom point has period 60. Since this is a lattice polygon, all orbits are periodic and this implies that the web in any bounded region will stabilize at some level- $k$ . In this region,  $k$  seems to be 51

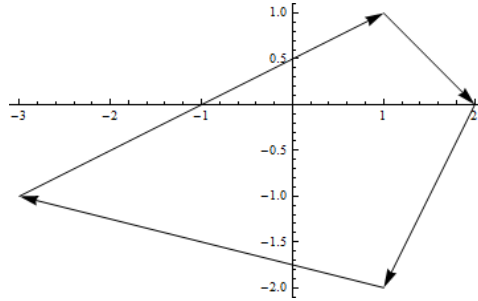


To obtain a 'corrected' red web for  $M$  is more complex than the regular case, because we cannot just apply  $R_t$  to the forward web to get the inverse web. We have to first invert  $M$  by applying  $R_t$ , then take the forward web of  $R_t[M]$  and apply  $R_t$  to the forward web. There is actually a 'conjugacy' between  $M$  and  $R_t[M]$  which we will define below.



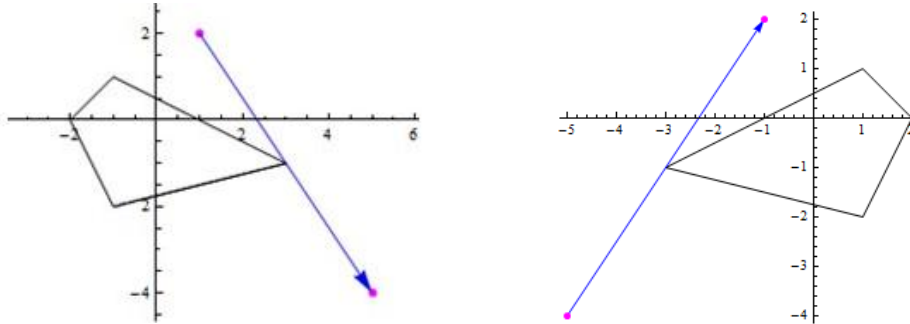
**Example:**  $Rt[M] = \{\{1,1\},\{-3,-1\},\{1,-2\},\{2,0\}\}$ . Mathematica does this operation in the same order that vertices are listed in M, but now the order of the vertices is counter-clockwise and we want to retain the clockwise rotation, so replace Rt with **RRt**[p\_]:= **Reverse**[Rt[p]];

**RRt**[M] =  $\{\{2,0\},\{1,-2\},\{-3,-1\},\{1,1\}\}$  as shown below

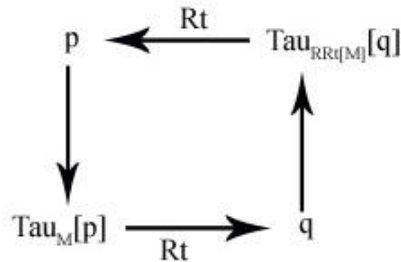


We call this polygon the 'reflection' of M. It has the 'same' dynamics as M but the new  $\tau$  is  $\tau^{-1}$  relative to M.

For example, using M, **Tau**[[1,2]] =  $-\{1,2\} + 2M[[2]] = \{5,-4\}$  as shown on the left below. On the right is the 'same' map in reverse: **ReflectionTransform**[[1,0]][{5,-4}] =  $\{-5,-4\}$  and this maps to **Tau**[-5,-4] =  $-\{-5,-4\} + 2RRt[M][[3]] = \{-1,2\}$  which is **ReflectionTransform**[[1,0]][{1,2}].

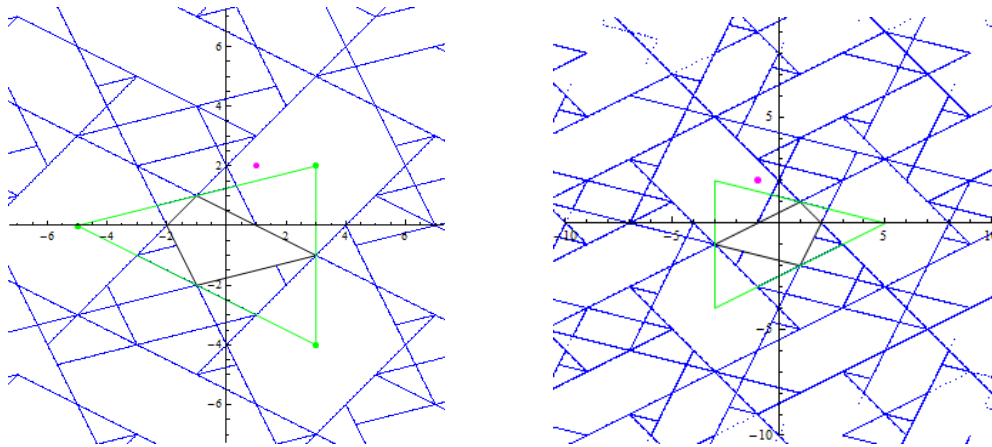


The following diagram commutes:



This implies that the two polygons have reflected dynamics and reflected webs.

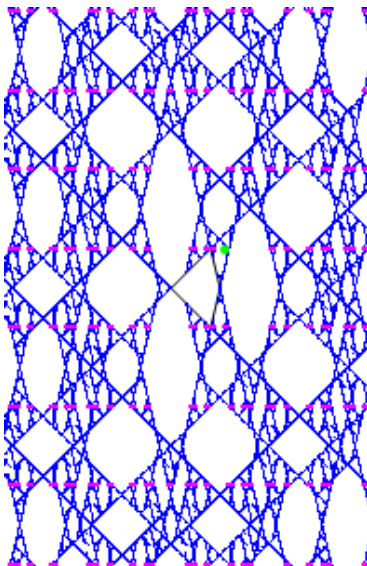
On the left below is the web for  $M$  showing the point  $\{1,2\}$  in magenta and the period 3 orbit of the point  $\{-5,0\}$  in green. On the right is the web for  $RRt[M]$  showing the matching point  $\{-1,2\}$  in magenta and the 'same' green orbit but with  $\{5,0\}$  as the initial point.



**Example:** The Penrose Kite

As pointed out earlier, the forward web is not 'wrong', but the user has to be aware that the regions it defines are relative to the reverse orientation. This would be an issue if someone picks out a region of the web and tries to identify **how** it evolved. In this case it would be wise to use the inverse web - especially if the region in question is not 'stable'. We will illustrate this below with the Penrose kite.

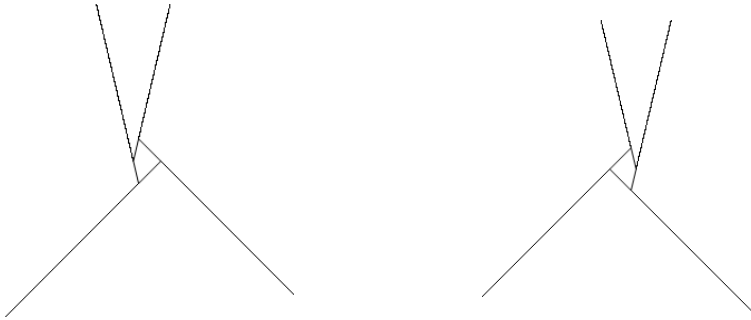
On the left below is the Penrose kite and its forward web. At this resolution it looks exactly like it is 'suppose' to look. On the right is a graphic made by Richard Schwartz for the *Math Book* by Clifford Pickover (see the section on the Penrose kite for details)



The green dot on the left is the point  $p = \{1/2 (3-\text{Sqrt}[5]), 1\}$ . This point has an unbounded orbit and that implies that the local web around  $p$  will always be evolving. The magenta 'thread' shown above are points in the orbit of  $p$ . They all lie on a lattice of lines of the form  $y = k$  for  $k$  odd, so  $p$  starts out on the line  $y = 1$  and branches out to the other odd lines.

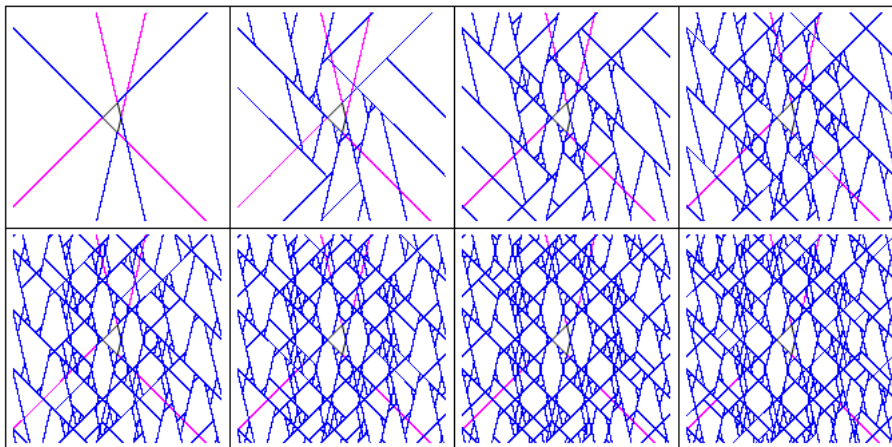
The problems with the forward web start to show up when we zoom in, but first we will see how the forward-trailing edge coverage compares to the earlier examples of the regular pentagon and the lattice polygon.

To generate the trailing edges, use the conjugate of the Penrose kite which just flips it about the  $y$  axis. This polygon is shown on the left below with its forward edges as generated by the web command. The right side is  $R_t$  applied to this web. These are the trailing edges of the Penrose kite. We will color them magenta and overlay the normal web on top as we did above with the regular pentagon and the lattice polygon.

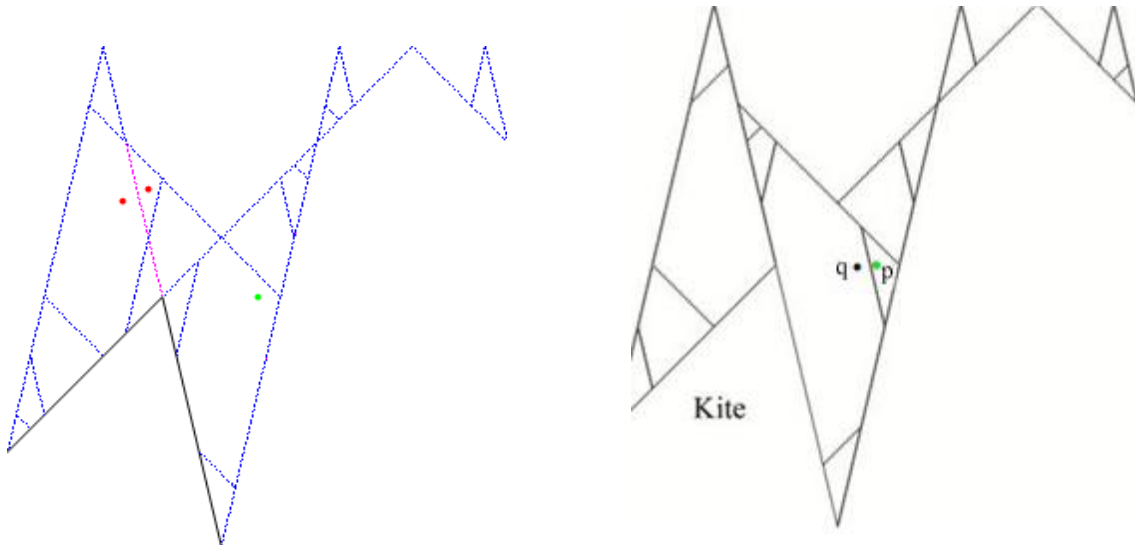


```
Gr[depth_]:=Show[Graphics[{AbsolutePointSize[1.0],poly[Mom],Magenta,Point[W1],Blue,Point[WebPoints[.01,12,depth]]},PlotRange->{{left,right},{bottom,top}}];
```

```
GraphicsGrid[{{Gr[0],Gr[3],Gr[6],Gr[9]},{Gr[12],Gr[15],Gr[18],Gr[21]}},Frame->All]
```



Compared to the regular case and the lattice polygoin, this is a poor match between these two webs. Below is the region around p (green dot) after 50 iterations. The forward web in this region has many errors. For example the two orange points should have been separated 'at-birth' by the magenta trailing edge but they are still grouped together by the blue web. This is a level-1 discrepancy but there are differences that appear at more advanced levels. For example there is a level-39 error which is illustrated on the right. There should be a dividing line just to the left of p.

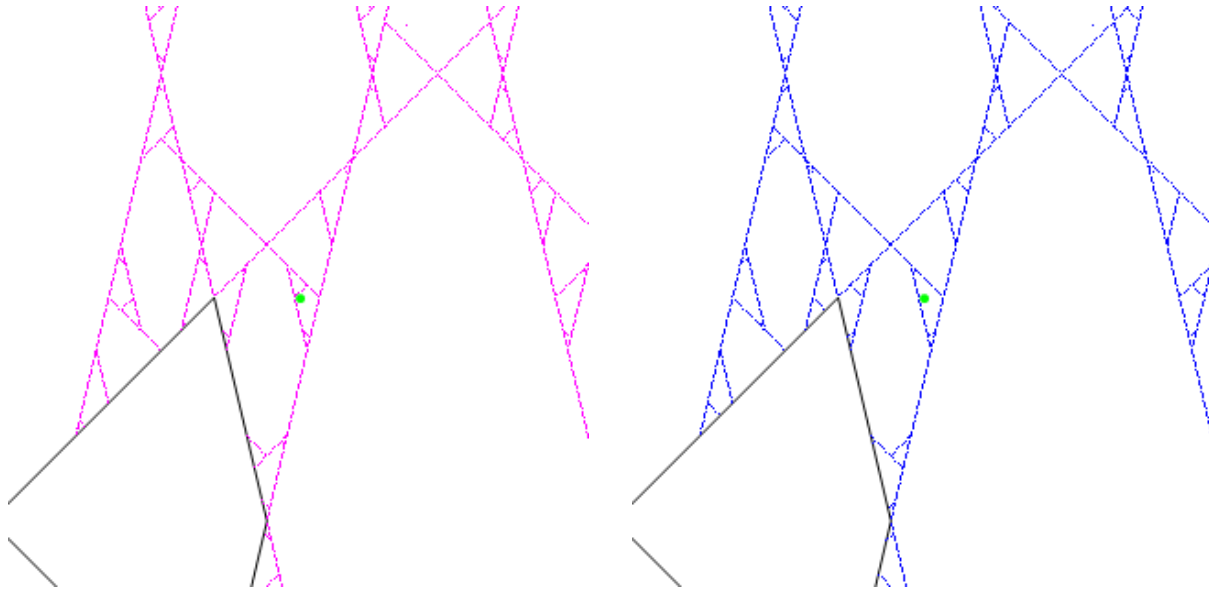


It is easy to determine exactly when this line should appear - just compare the corner sequences or step sequences of p and q. These step sequences are shown below. They are identical until the 39th place and therefore the boundary line between them is part of the level-39 (inverse) web but not the level-38 (inverse) web. On the magenta web this is true, but it is clearly false on the blue web.

Step sequence for p:  $V_p[p,50] = 1, 1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 1, 3, 1, 2, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1, 3, 1, 2, 2, 1, \dots$

Step sequence for q:  $V_p[q,50] = 1, 1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 1, 3, 1, 2, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1, 3, 1, 2, 2, \mathbf{2}, \dots$

Below is the inverse web to level 1200 in magenta and the corresponding forward web in blue. After 1200 iterations many differences still persist, but they are on a smaller scale so we have plotted them side by side below.



The blue forward web contains the same amount of detail, but it is detail about the inverse map  $\sigma$ . These are eventual boundary points of the map  $\tau$ , so we will define the web of a polygon to be the union of the inverse and forward webs.

**Definition:** For a convex polygon  $M$ , let  $W_0$  be the union of the forward and trailing extensions of the edges of  $M$ .

(i) The level  $k$  **forward web** of  $M$  is defined to be  $\bigcup_{j=0}^{j=k} \tau^j(W_0)$

(ii) The level  $k$  **inverse web** of  $M$  is defined to be  $\bigcup_{j=0}^{j=k} \tau^{-j}(W_0)$

(iii) The (limiting) **web** of  $M$  is defined to be  $\bigcup_{j=-\infty}^{j=\infty} \tau^j(W_0)$

In any bounded region containing the origin, the web may 'stabilize' after a certain number of iterations and in this case the forward web and the inverse web will be identical and all subdomains will be periodic. This will occur with any lattice polygon.

For regular (non-lattice) polygons the webs will never stabilize in any region of interest. The two webs will have the 'same' information at every iteration, but only the inverse web will match the development of the subdomains. After a few hundred iterations, the two webs will seem to be indistinguishable, but there will always be differences on some scale. For non-regular polygons such as the Penrose kite, the two webs are not symmetric and one cannot be inferred from the other. The two webs may still show noticeable differences after thousands of iterations.

Summary of periodicity of orbits for the Tangent Map for regular polygons

N-gon	Periods
N: Prime	The canonical 2Ngons $S[1], S[2], \dots, S[\text{Half}N]$ all have period N (mapping centers) The canonical Dad step-k has period $N(N-(k+2))$ . For example Mom[1] is Dad step-1 so she has period $N(N-3)$ and Dad[1] is step-2 with period $N(N-4)$ .
N: Non prime odd	The canonical periods are the same as above, but the prime periods may be smaller due to decomposition of orbits. For example with $N = 9$ , Helen3 is Step-3 of Dad so she should have period $9(9-5) = 36$ but the prime period is 12 and there are three groups of 12 to make up the 36.
N: $4k+1$ prime	For super-symmetric $(4k+1)$ primes there appear to be Moms and Dads at all generation scales and the ratio of periods $\text{Dad}[k]/\text{Dad}[k-1] \rightarrow N+1$ This limiting ratio applies equally to the chain of Mom's.
N = 5	<p>If <math>d[n]</math> is the period of decagons and <math>p[n]</math> is the period of pentagons, then</p> <p><b><math>d[n] := (5/7)(8*6^{(n-1)} + (-1)^n)</math>;</b>  <b><math>p[n] := 5*((3*d[n]/5)+(-1)^n)/2+(-1)^{(n-1)}</math>;</b></p> <p>For example: <b><math>d[\text{Range}[10]] = \{5, 35, 205, 1235, 7405, 44435, 266605, 1599635, 9597805, 57586835\}</math></b>  <b><math>p[\text{Range}[10]] = \{10, 50, 310, 1850, 11110, 66650, 399910, 2399450, 14396710, 86380250\}</math></b></p> <p>We can easily translate these values to the rings of Dads so this is the only non-trivial case where all the periods are known.</p>
N = 7	<p>Even though 7 is a <math>4k+3</math> prime it does support infinite chains of Dads and Moms converging to the GenStar point, but the even and odd families have different dynamics. If 7 was a <math>4k+1</math> prime, the ratio of these periods would approach <math>N + 1</math>, but here there are two ratios which seem to be 8 and 25, so the ratio for even and odd generations appears to approach 200. The Moms periods for generation k are dependent on the Dads periods for generation k-1, because the canonical Mom[k]'s are on the edges of the Dad[k-1]'s. The exact nature of this dependency is not known.</p> <p>There are also infinite sequences of buds at the 'qstar' point at the border between S2 and Helen. This is also the border between the inner and outer star regions. The converging buds here are S2's and Portal Moms. In both cases the ratio of periods appears to approach 113.</p> <p>In the inner star region, the Star[2] point has similar dynamics but here the dichotomy is between S1 buds and S2 buds. The S1 family periods are 7, 322, then bifurcations into families of 827 and 1071 and these bifurcations continue. S2 periods are 7,46 also followed by bifurcations. No formula is known for S1 or S2 periods, but it appears that the ratios approach 113, the same as qstar. Note that 113 'factors' as 9 and 12.5 for odd-even vs. even-odd</p>

N = 8,9,12	Even though these are not prime, they support infinite strings of Dads and Moms and in each case the ratio of periods $Dad[k]/Dad[k-1] \rightarrow N+1$ . In $N = 8$ and $N = 12$ these Dads are dense so they can be used to find the fractal dimension.
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Based on the cases for  $N = 9$  and  $N = 12$  it seems likely that the  $N+1$  rule applies to all polygons which support infinite chains of Dads and Moms converging to the GenStar point. There may be related rules which apply to convergence at the other star points.

### References

F. Vivaldi, A. Shaidenko, *Global stability of a class of discontinuous dual billiards*. Comm.Math. Phys. **110**, 625–640. (1987) MR895220 (89c:58067)