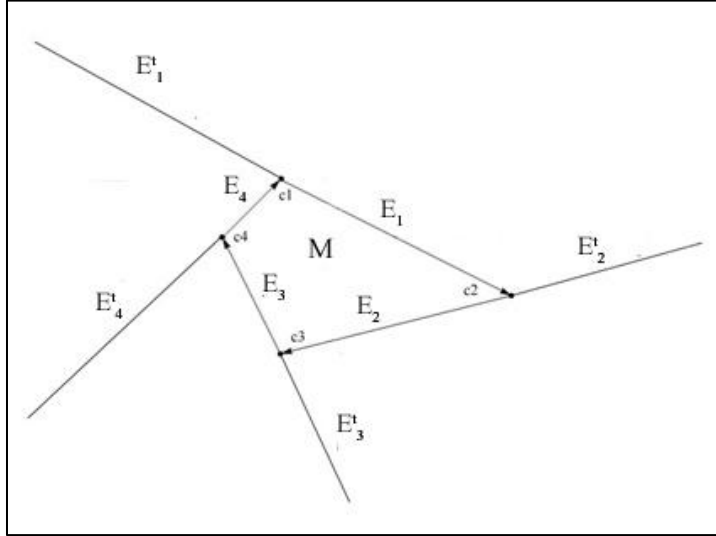


DynamicsOfPolygons.org

The web of a convex polygon

Definition of the Web for a convex polygon M

Given a convex n -gon M with edges E_1, E_2, \dots, E_n , each edge has an orientation which we assume to be clockwise by default. Therefore each edge has a 'forward' and 'trailing' extension. Let E_k^f be the forward extension of edge E_k , and similarly E_k^t is the trailing extension of edge E_k .



The Tangent map is τ not defined on the trailing edges, so the level 0 web is defined to be $W_0 = \bigcup_{k=1}^{k=n} E_k^f$. The complement of W_0 external to M consists of n disjoint open sets which we call the level 0 subdomains. The union of these n subdomains is the domain of τ which we abbreviate $\text{Dom}(\tau)$. $\text{Dom}(\tau^2)$ is $\text{Dom}(\tau) - \sigma(W[0])$ where $\sigma = \tau^{-1}$. The union of $W[0]$ and $\sigma(W[0])$ is called the level 1 web, W_1 . In general the level k (inverse) web is defined to be:

$$(i) \quad W_k^i = \bigcup_{j=0}^{j=k} \tau^{-j}(W_0) \quad \text{where} \quad W_0 = \bigcup_{k=1}^{k=n} E_k^f$$

The forward web is defined in a similar fashion using the extended forward edges:

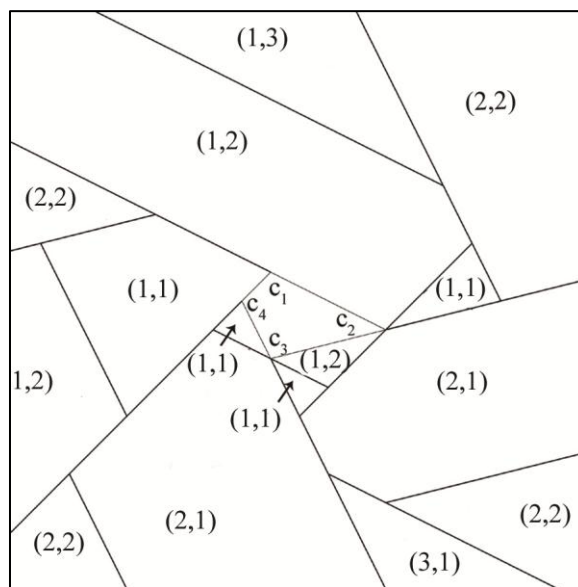
$$(ii) \quad W_k^f = \bigcup_{j=0}^{j=k} \tau^j(W_0) \quad \text{where} \quad W_0 = \bigcup_{k=1}^{k=n} E_k^t$$

The level k web W_k is the union of these two webs, and the limiting web W is $\lim_{k \rightarrow \infty} W_k$. The web W for a convex polygon M is the *genetic structure* of the polygon relative to the map τ . The web is also known as the 'exceptional' set of τ . It plays a role similar to the Julia set for complex mappings. The complement W^c consists of the points in the plane where τ^k is defined for all integers k , so W^c is $\lim_{k \rightarrow \pm\infty} \text{Dom}(\tau^k)$.

At every iteration, W_k (together with the edges of M) is connected so the complement will consist of disjoint open convex sets, some of which are always unbounded. These are the level- k ‘subdomains’ which are also called ‘tiles’ or ‘regions’. For regular polygons these regions are sometimes called ‘buds’ because they tend to form on the perimeter of previous regions.

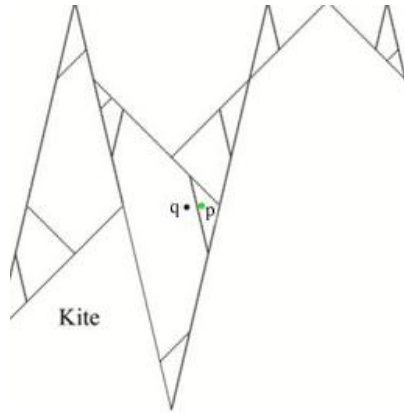
Every subdomain at level k can be labeled with a corner sequence which lists the vertices visited by any point in that region. The first differences in these corner sequences yields a step sequence. If another subdomain is part of this same orbit, it will have the same step sequence except for a displacement – so for periodic orbits, the step sequences will also be periodic, but the prime period could be shorter.

Example: Below is the level 2 web for the polygon M from above, showing the step sequences



Example: Below is a portion of the (inverse) web for the Penrose Kite. The point p shown in green has an unbounded orbit and its limiting subdomain will be a single point. However there will be points arbitrarily close to p which belong to the same unbounded orbit and hence have the same step sequence. These points will form a horizontal Cantor set.

The short web segment separating p and q below was not part of the level-38 (inverse) web, but it is part of the level 39-th web, which is the web shown here. This means that these two points had identical step sequences until the 39th iteration.



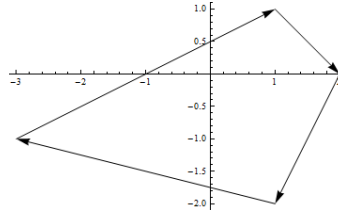
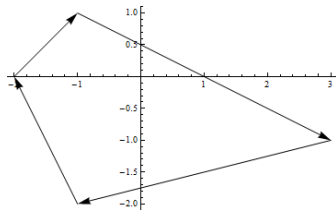
Step sequence for p: $Vp[p,40] = 1, 1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 1, 3, 1, 2, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1, 3, 1, 2, 2, 1, \dots$

Step sequence for q: $Vp[q,40] = 1, 1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 1, 3, 1, 2, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1, 3, 1, 2, 2, \mathbf{2}, \dots$

All the points in a given subdomain have the same corner sequence so the dynamics of points can be replaced with the dynamics of regions. As indicated above, some of these subdomains may have limits which are points or line segments. For example the regular pentagon has subdomains with arbitrarily small Lebesgue measure – and arbitrarily large periods. It is likely that all regular n-gons for $n > 6$ have structure on all scales – which implies there will be subdomains with arbitrarily small Lebesgue measure. The web itself has measure zero but it may have fractal dimension greater than 1.

For any polygon M, the inverse map $\sigma = \tau^{-1}$, can be obtained by reversing the orientation of the edges from clockwise to counter-clockwise. From a practical standpoint, this can be avoided by inverting the polygon and retaining a clockwise orientation, so that τ is unchanged

Definition: For any polygon M, $Rt[M]$ is defined to be the reflection of M about the y-axis. In Mathematica: $Rt[M_]:=ReflectionTransform[\{1,0\}][M]$. The two polygons below are reflections of each other. Note that the clockwise orientation has been preserved.



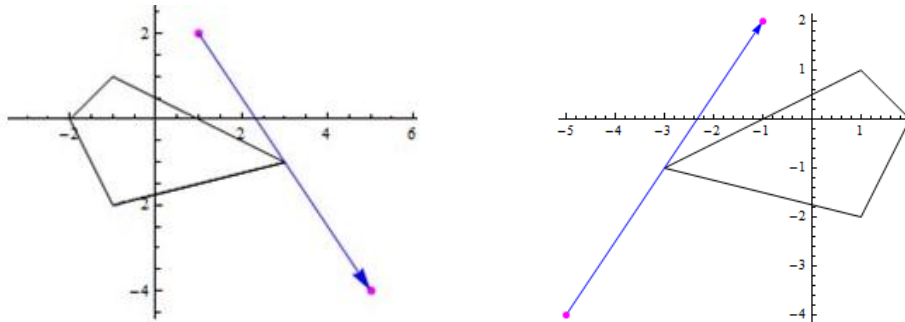
Mathematica does this operation in the same order that the vertices are listed in M, but now the order of the vertices is counter-clockwise and we want to retain the clockwise rotation, so replace Rt with $RRt[p_]:=Reverse[Rt[p]]$; We will therefore define M^t to be $RRt[M]$.

In terms of vertices, the example on the left above has $M = \{-1, 1\}, \{3, -1\}, \{-1, -2\}, \{-2, 0\}$

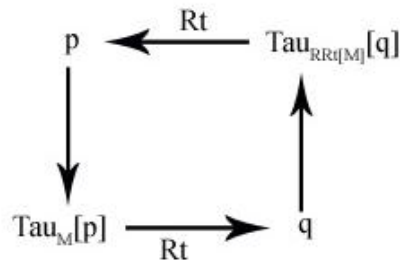
So $Rt[M] = \{1, 1\}, \{-3, -1\}, \{1, -2\}, \{2, 0\}$ and $M^t = RRt[M] = \{2, 0\}, \{1, -2\}, \{-3, -1\}, \{1, 1\}$

We call M^t the 'reflection' of M . It has the 'same' dynamics as M but the new τ is τ^{-1} relative to M .

Example: Using M from above, $\mathbf{Tau}[\{1, 2\}] = -\{1, 2\} + 2M[[2]] = \{5, -4\}$ as shown on the left below. On the right is the 'same' map in reverse: $\mathbf{ReflectionTransform}[\{1, 0\}][\{5, -4\}] = \{-5, -4\}$ and this maps to $\mathbf{Tau}[\{-5, -4\}] = -\{-5, -4\} + 2RRt[M][[3]] = \{-1, 2\}$ which is $\mathbf{ReflectionTransform}[\{1, 0\}][\{1, 2\}]$.



There is a simple conjugacy between the dynamics of M and the dynamics of M^t . The following diagram commutes:



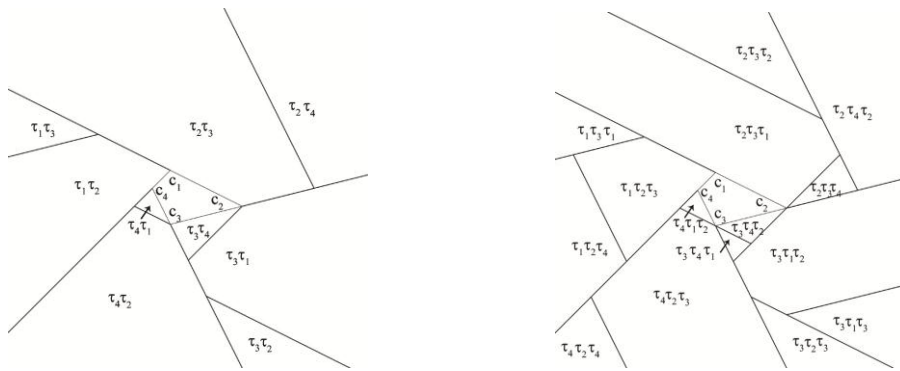
So the two polygons have reflected dynamics and reflected webs. This means that we can obtain the inverse web of a regular polygon by simply inverting the forward web. This can be done in a matter of seconds even for a web with millions of points. For non-regular polygons which are not symmetric about the y axis, we have to use M^t and generate the corresponding forward web and then invert this web and map it back to M to get the inverse web. We will give examples below and show the efficacy of using inverse webs in conjunction with forward webs.

In the limit, the forward web and inverse web for any convex polygon should be the same, but this limit may be a very long time off, so there are many reasons why an investigator would want to use both of them.

The Tangent Map software will generate the forward webs to any desired accuracy and a reflection transform will yield the inverse webs when necessary. The complement of this web are points where τ is always defined. For the regular triangle, this web consists of rings of congruent triangles and hexagons as shown on the left. This is what we call the genetic structure of the regular triangle. Any other triangle would have a similar web because every triangle is an affine transformation of the equilateral triangle. Run [CDF Web Demo](#).

Example: The lattice polygon: $M = \{-1, 1\}, \{3, -1\}, \{-1, -2\}, \{-2, 0\}$ is shown below with $M[[1]] = c1 = \{-1, 1\}$. The edges are numbered clockwise. The Tangent Map reduces to four maps. The 'trailing' edges $E^C_1, E^C_2, E^C_3, E^C_4$ form the level 0 exceptional set W . If we include the edges of M , then W is connected. This is called the level 0 web, $W[0]$. These are the only points external to M , where τ is not defined. The complement of the web consists of disjoint open regions which we call the level 1 domains. The four level 1 domains for M are shown below.

Since τ is not defined on $W[0]$, τ^2 is not defined on the inverse image of $W[0]$ so $\text{Dom}(\tau^2) = \text{Dom}(\tau) - \sigma(W[0])$ where $\sigma = \tau^{-1}$. The union of $W[0]$ and $\sigma(W[0])$ is called the level 1 web as shown on the left below. The level k web is $W[k] = \text{NestList}[\sigma, W[0], k]$. This is a 'working definition', formal definitions will follow.

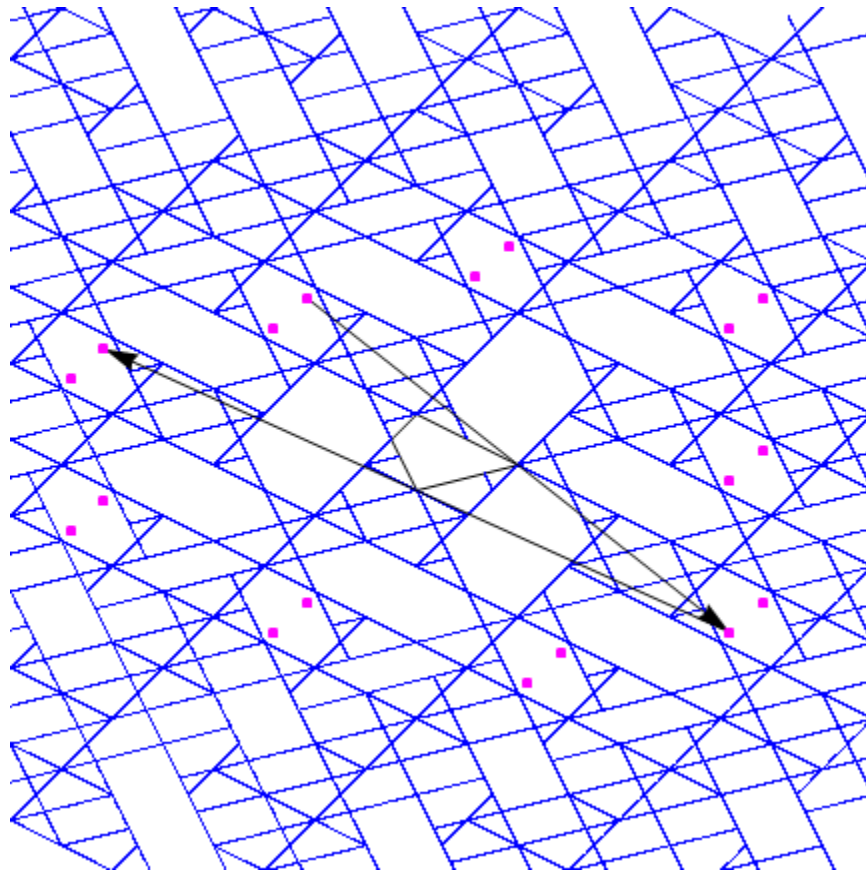


Each region in the complement of the web can be labeled with the corresponding 'corner sequence' and these in turn can be used to classify the dynamics. Any level k subdomain will map undivided for at least k iterations of τ , so we can define periods of subdomains in the same way that we define periods of points.

Below is a portion of the level 300 web for M , showing 9 congruent subdomains which map to each other in a periodic fashion. Since τ is an inversion, the points return to the original region after 9 inversions. Because this is odd, these points will have to make two circuits, and their period will be 18. But there is a central point which feels no inversions, so the center has period 9. This scenario is called 'period doubling' and it is generic for all regions which have odd period.

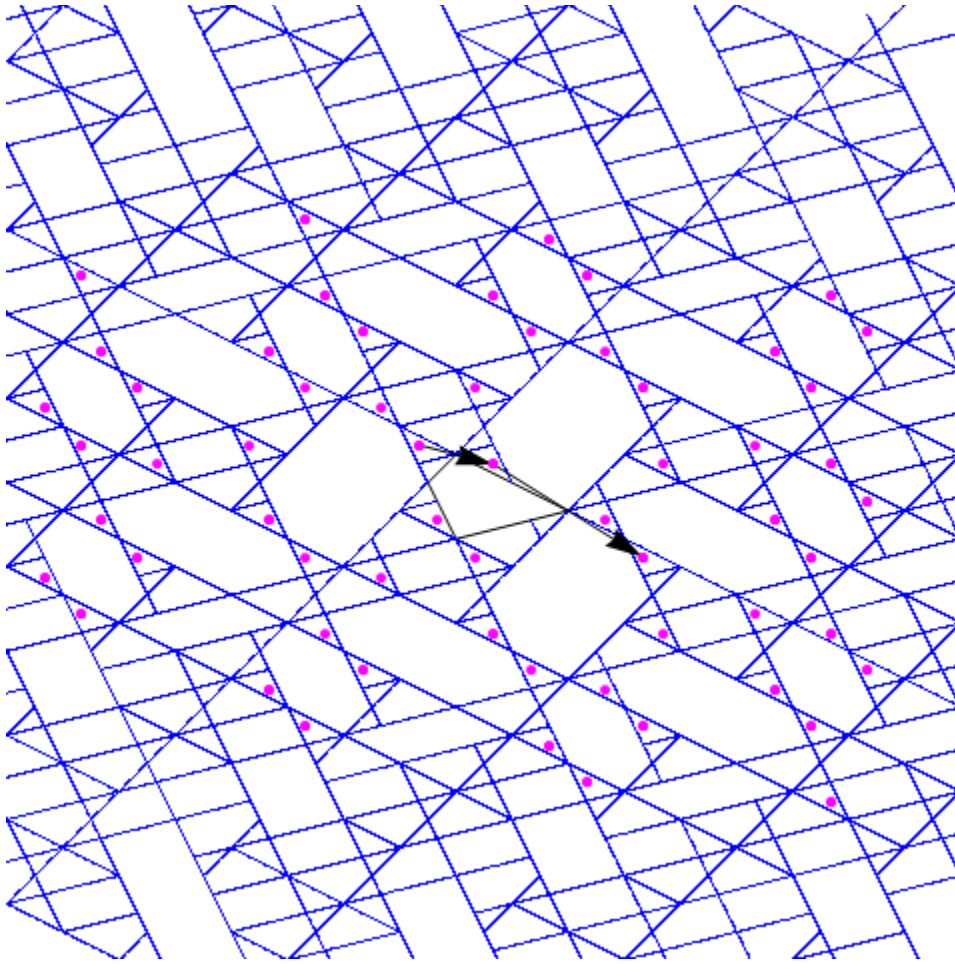
Our convention is to call this a period 9 ring. By the same convention, the ring of three large 'Dads' below, would be called a period 3 ring, even though all points are period 6, except for the center.

q1= {-5.333,5.583}; K = V[q1,200]; Period[K] Out = 18.



The corner sequence for this orbit is $\mathbf{IND}[\mathbf{q1}, \mathbf{18}] = \{3, 4, 2, 4, 2, 3, 1, 3, 4, 3, 4, 2, 4, 2, 3, 1, 3, 4\}$. Note that it is period 9 with step sequence $V_p[\mathbf{q1}, 9] = \{1, 2, 2, 2, 1, 2, 2, 1, 3\}$.

If a subdomain has even period then any point will return after an even number of inversions. Since the entire subdomain must map to itself, this mapping must be the identity. Therefore every point has the same even period. For example using M from above $\mathbf{q1} = \{-2.345, 1.353\}$ has period 60 and every other point in this region has the same period, so there are 60 congruent regions as shown below.



The corner sequence and step sequence are period 60. Below is the step sequence: $V_p[q1,60]$
 $\{1,1,2,1,2,2,1,2,2,2,1,2,2,1,3,1,2,2,2,1,2,2,2,2,1,2,2,2,2,2,1,2,2,1,3,1,2,2,2,1,2,2,2,1,1,3,1,2,$
 $1,2,2,1,2,1,1,1,1\}$

Lemma: If a subdomain has even period k , then all points in the subdomain have period k . If the subdomain has odd period k , the all points in the subdomain have period $2k$ except for the center which has period k .

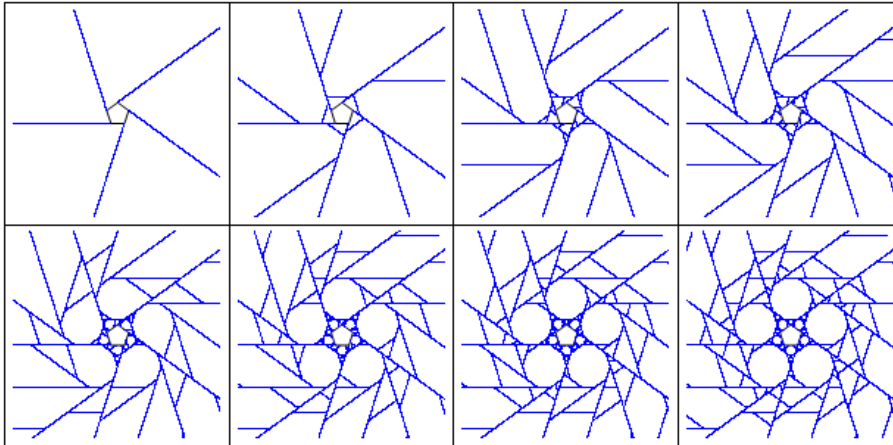
From a practical standpoint it is easier to generate the forward web because it uses the original tangent map τ . All the notebook packages generate forward webs. To generate a true inverse webs, we would need to implement the inverse map σ , which is the same as τ but with a counter-clockwise orientation. For regular N -gons this counter-clockwise version of the polygon is identical to the original so we can generate a forward web and imagine that we are working with a polygon that originally had counter-clockwise orientation.

All the notebooks share the same web algorithm: `web[stepsize_,length_,depth_,offset_]` which scans the forward edges at a given step size out to the given length and generates k images of each point where $k = \text{depth}$. The routine called `WebPoints` collects these edge scans together into a file for plotting.

Example: The regular pentagon $N = 5$

```
Gr[depth_] := Show[Graphics[{AbsolutePointSize[1.0], poly[Mom], Blue,
  Point[WebPoints[.01, 12, depth]]}, PlotRange -> {{left, right}, {bottom, top}}];
```

```
GraphicsGrid[{{Gr[0],Gr[1],Gr[2],Gr[3]},{Gr[4],Gr[5],Gr[6],Gr[7]}},Frame->All]
```

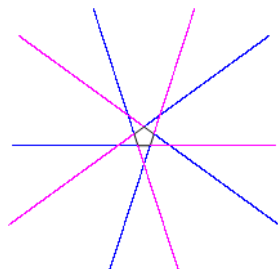


With some practice it is possible to track the evolution of the webs by swapping domains and ranges. This is called cobwbbing. For a function of one variable, the line $y = x$ performs this magic. For $N = 5$, by level-3 the inner edges of Dad have sealed off the star region because the 3rd iteration aligns with a trailing edge of Mom. All prime N -gons have invariant star regions after $N-2$ iterations.

There are times when it is useful to generate the true inverse web of a polygon. For example to associate regions with their corner sequences, the forward web will not work, because only the inverse web can distinguish subdomains properly at each iteration. Of course once regions are periodic the two webs yield the same information. In the interim it may be useful to generate both webs and take their union. We will do this with non-regular polygons below.

Example: Using the regular pentagon. $W0 = \text{WebPoints}[\mathbf{.01, 12, 0}]$. This will generate the depth 0 (Forward) web, which consists of the forward edges (as shown in blue below). To generate the backwards edges, which make up the depth 0 Inverse web, use $\text{Rt}[p_] := \text{ReflectionTransform}[\mathbf{\{1,0\}}][p]$; This will reflect points about the normal to $\{1,0\}$ which is the y -axis, so $\text{Rt}[W0]$ will yield the trailing edges as shown below in red.

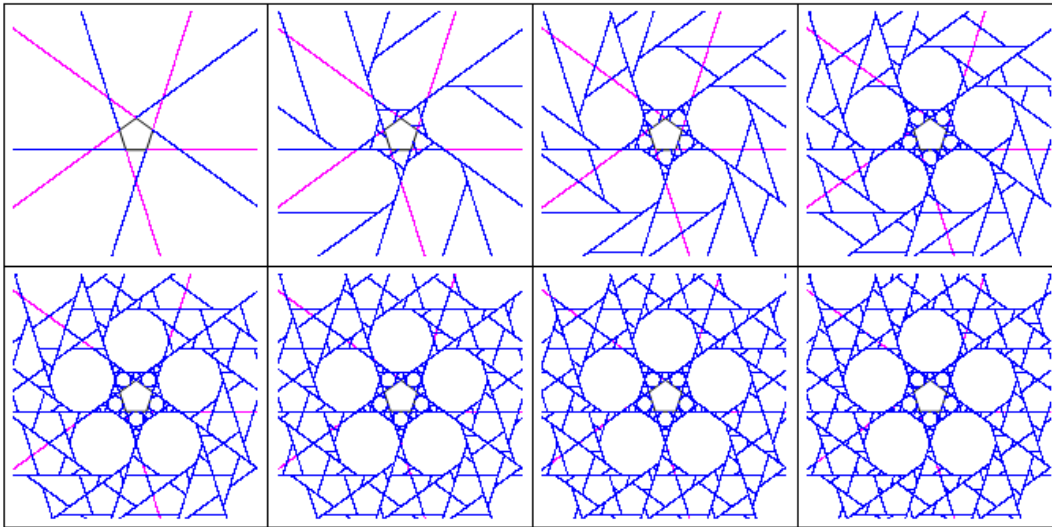
```
Graphics[{{poly[Mom],AbsolutePointSize[1.0], Blue, Point[W1], Red, Point[Rt[W0]]}]
```



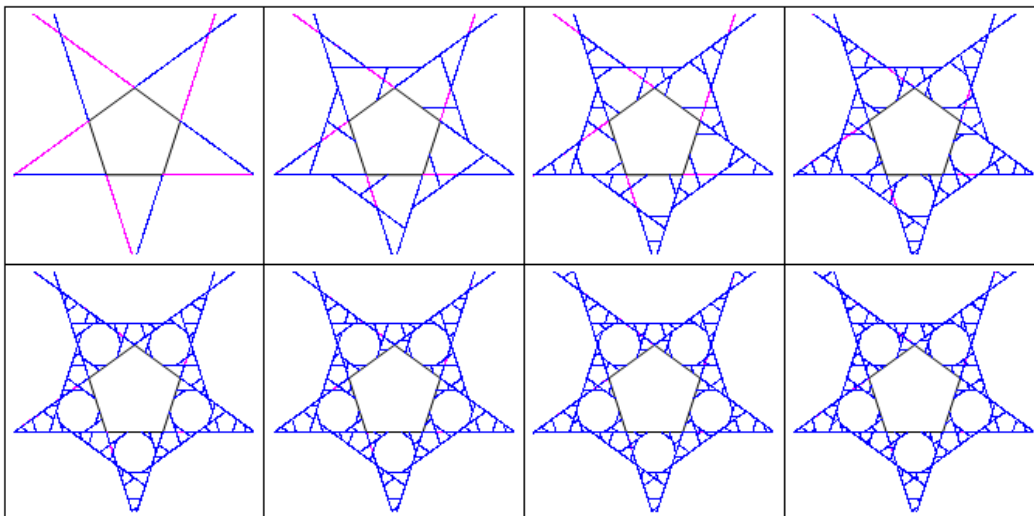
The blue edges can only map to the red and conversely. In the plots below we will map the blue (Forward) edges and keep the red edges fixed.

```
Gr[depth_]:=Show[Graphics[{AbsolutePointSize[1.0],poly[Mom],Magenta,Point[Rt[W]],
Blue,Point[WebPoints[.01,12,depth]]},PlotRange->{{left,right},{bottom,top}}];
```

```
GraphicsGrid[{{Gr[0],Gr[3],Gr[6],Gr[9]},{Gr[12],Gr[15],Gr[18],Gr[21]}},Frame->All]
```



By level 21 most of the forward edges have mapped to trailing edges, but there will always be 'hot' points because the web is always evolving - even in the central star region. Below is a close up of this evolution. It is no surprise that the 'hot' spots are converging to the non-periodic limit points.

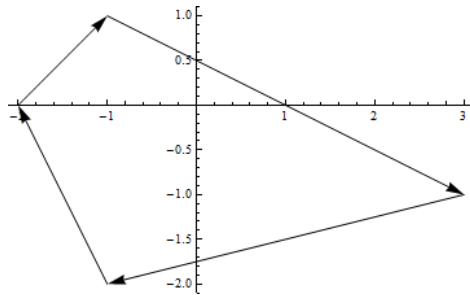


From a practical standpoint, the forward web of a regular polygon will give correct dynamics after just a few iterations. The webs are just visual aids and none of the code depends on the web. The forward webs are not 'incorrect' but the user should understand what information they

contain, and that information is the dynamics of the inverse map τ^{-1} . For periodic regions this is the same information as τ , but for non-periodic regions there will always be differences between the forward and trailing webs. We will show some examples with the Penrose kite.

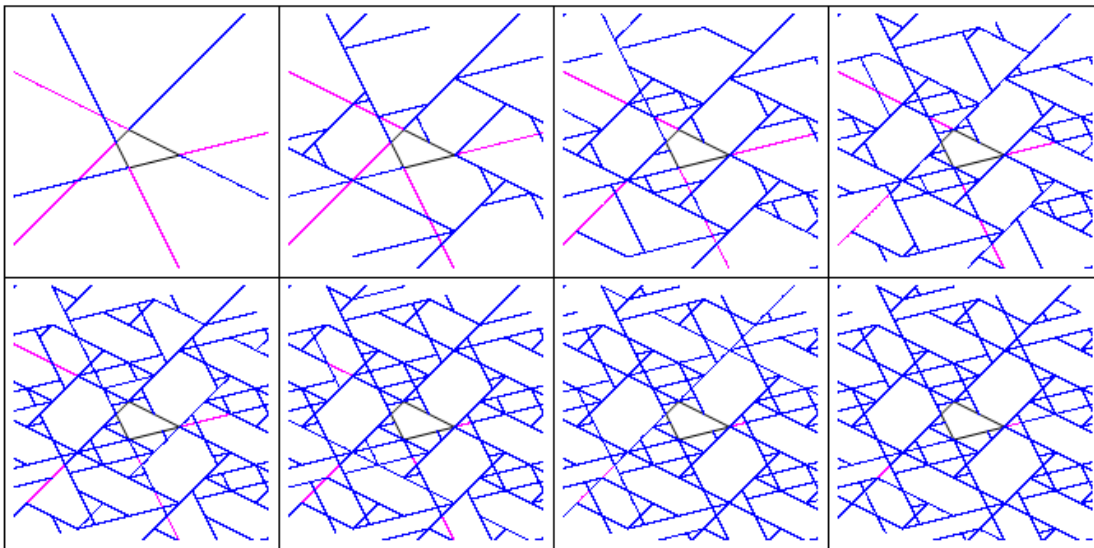
Webs of non-regular polygons

To see what happens for a typical non-regular polygon, we will use the lattice polygon defined above: $M = \{ \{-1, 1\}, \{3, -1\}, \{-1, -2\}, \{-2, 0\} \}$

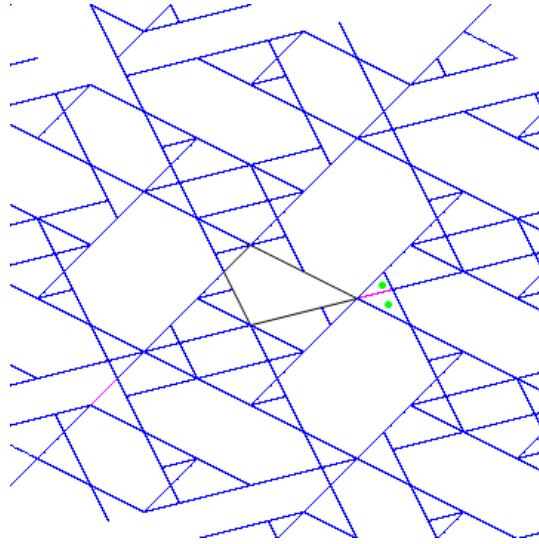


The reflection transformation R_t from above works for any polygon and we have applied it here to obtain the trailing edges shown below in red. We will once again map the blue forward edges using every third iteration:

GraphicsGrid[{{Gr[0],Gr[3],Gr[6],Gr[9]},{Gr[12],Gr[15],Gr[18],Gr[21]}},Frame->All]



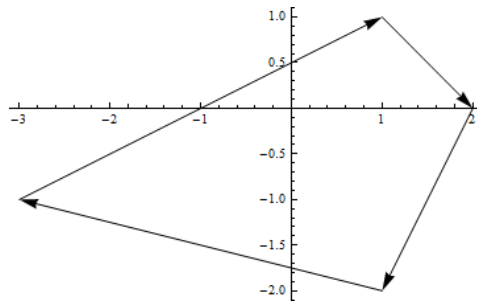
Below is an enlargement of the level 21 web from above. It should be clear that the two green points have different orbits from the very first iteration, but after 21 iterations it still looks like they are related. It is not until the level 51 web that this region is partitioned correctly. The trailing edges partitioned it correctly on the first iteration. The top point has period 154 and the bottom point has period 60. Since this is a lattice polygon, all orbits are periodic and this implies that the web in any bounded region will stabilize at some level- k . In this region, k seems to be 51



To obtain a 'corrected' red web for M is more complex than the regular case, because we cannot just apply Rt to the forward web to get the inverse web. We have to first invert M by applying Rt , then take the forward web of $Rt[M]$ and apply Rt to the forward web. As indicated earlier there is a 'conjugacy' between M and $Rt[M]$.

Example: $Rt[M] = \{\{1,1\},\{-3,-1\},\{1,-2\},\{2,0\}\}$. Mathematica does this operation in the same order that vertices are listed in M , but now the order of the vertices is counter-clockwise and we want to retain the clockwise rotation, so replace Rt with **$RRt[p_]:=Reverse[Rt[p]]$** ;

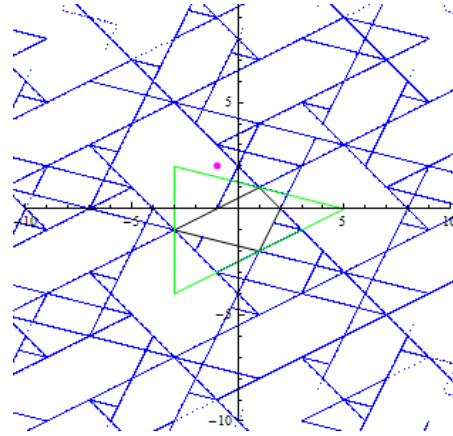
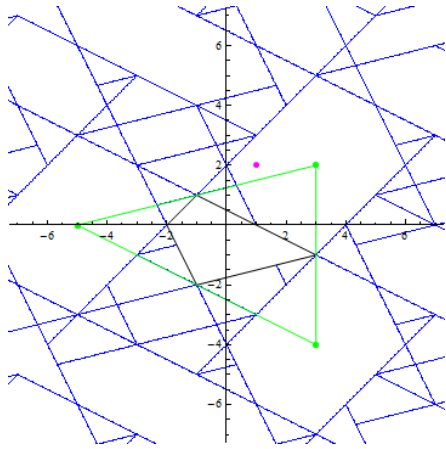
$RRt[M] = \{\{2,0\},\{1,-2\},\{-3,-1\},\{1,1\}\}$ as shown below



We call this polygon the 'reflection' of M . It has the 'same' dynamics as M but the new τ is τ^{-1} relative to M .

This implies that the two polygons have reflected dynamics and reflected webs.

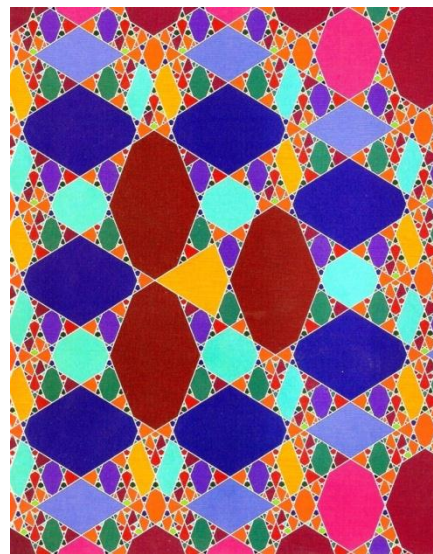
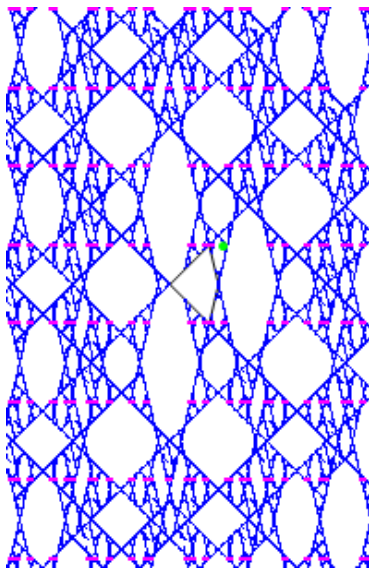
On the left below is the web for M showing the point $\{1,2\}$ in magenta and the period 3 orbit of the point $\{-5,0\}$ in green. On the right is the web for $RRt[M]$ showing the matching point $\{-1,2\}$ in magenta and the 'same' green orbit but with $\{5,0\}$ as the initial point.



Example: The Penrose Kite

As pointed out earlier, the forward web is not 'wrong', but the user has to be aware that the regions it defines are relative to the reverse orientation. This would be an issue if someone picks out a region of the web and tries to identify **how** it evolved. In this case it would be wise to use the inverse web - especially if the region in question is not 'stable'. We will illustrate this below with the Penrose kite.

On the left below is the Penrose kite and its forward web. At this resolution it looks exactly like it is 'suppose' to look. On the right is a graphic made by Richard Schwartz for the *Math Book* by Clifford Pickover (see the section on the Penrose kite for details)

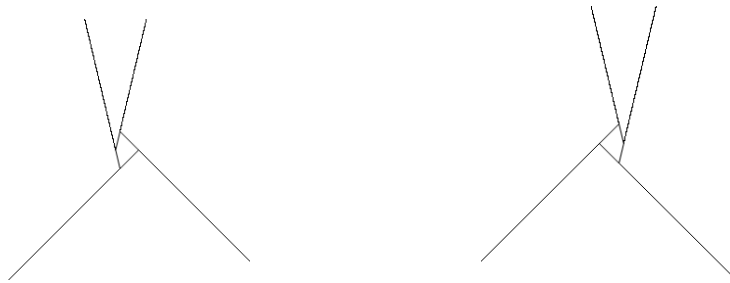


The green dot on the left is the point $p = \{ \frac{1}{2} (3 - \sqrt{5}), 1 \}$. This point has an unbounded orbit and that implies that the local web around p will always be evolving. The magenta 'thread' shown

above are points in the orbit of p . They all lie on a lattice of lines of the form $y = k$ for k odd, so p starts out on the line $y = 1$ and branches out to the other odd lines.

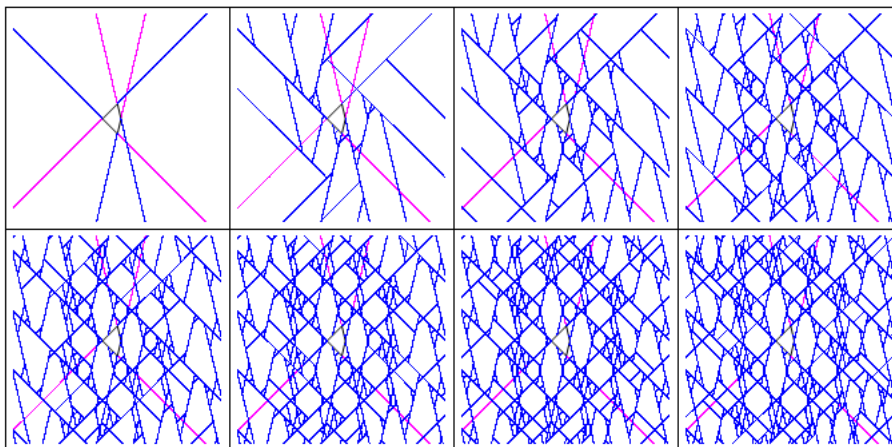
The problems with the forward web start to show up when we zoom in, but first we will see how the forward-trailing edge coverage compares to the earlier examples of the regular pentagon and the lattice polygon.

To generate the trailing edges, use the conjugate of the Penrose kite which just flips it about the y axis. This polygon is shown on the left below with its forward edges as generated by the web command. The right side is R_t applied to this web. These are the trailing edges of the Penrose kite. We will color them magenta and overlay the normal web on top as we did above with the regular pentagon and the lattice polygon.



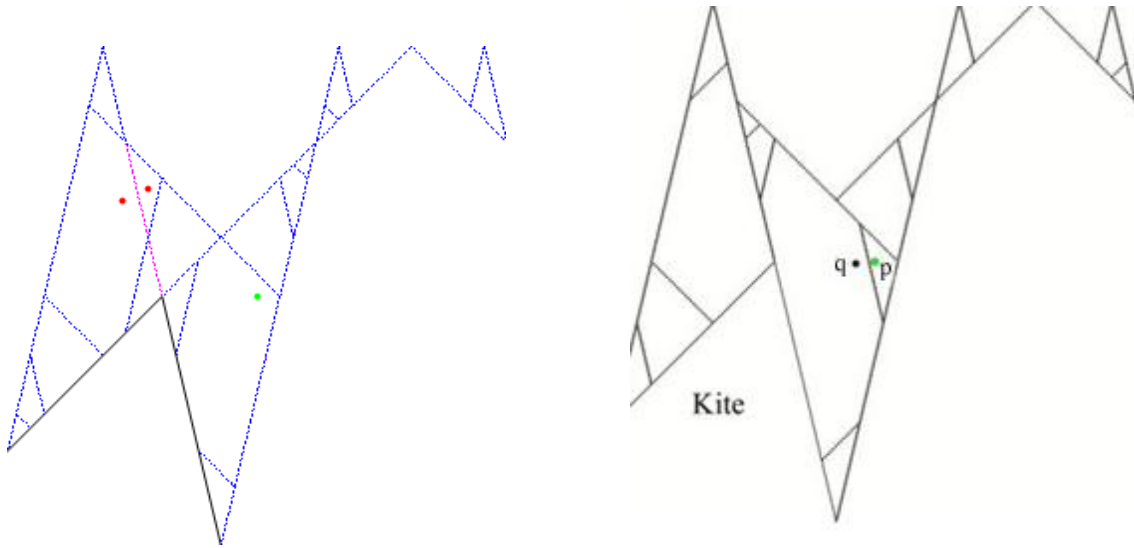
```
Gr[depth_]:=Show[Graphics[{AbsolutePointSize[1.0],poly[Mom],Magenta,Point[W1],Blue,Point[WebPoints[.01,12,depth]]},PlotRange->{{left,right},{bottom,top}}];
```

```
GraphicsGrid[{{Gr[0],Gr[3],Gr[6],Gr[9]},{Gr[12],Gr[15],Gr[18],Gr[21]}},Frame->All]
```



Compared to the regular case and the lattice polygon, this is a poor match between these two webs. Below is the region around p (green dot) after 50 iterations. The forward web in this region has many errors. For example the two orange points should have been separated 'at-birth' by the magenta trailing edge but they are still grouped together by the blue web. This is a level-1 discrepancy but there are differences that appear at more advanced levels. For example there is a

level-39 error which is illustrated on the right. There should be a dividing line just to the left of p.

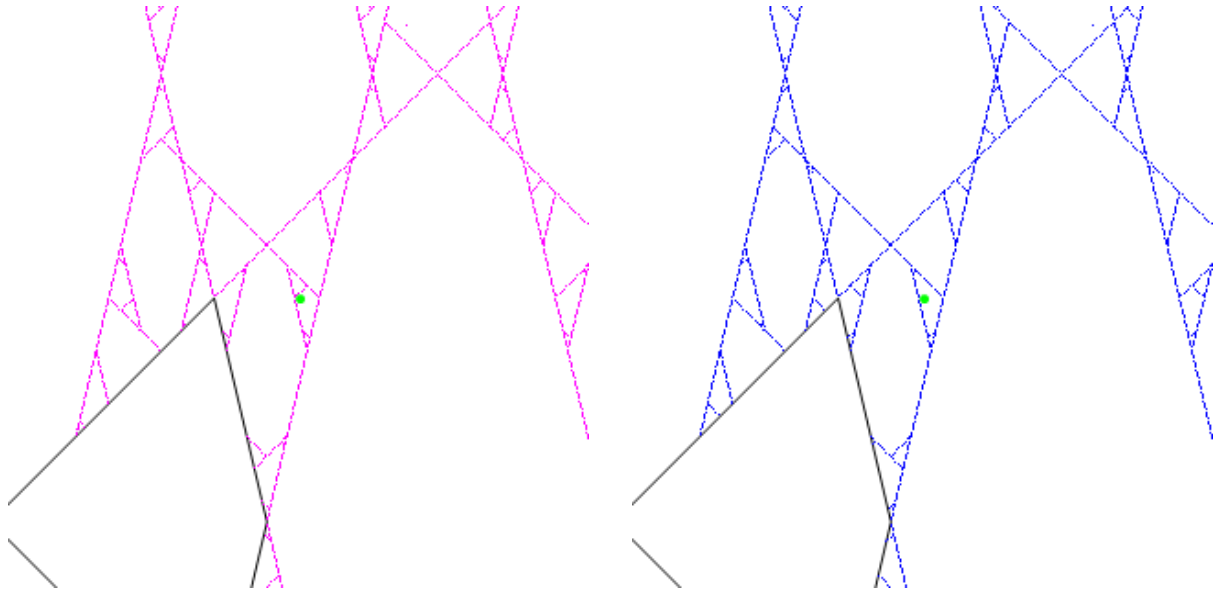


It is easy to determine exactly when this line should appear - just compare the corner sequences or step sequences of p and q. These step sequences are shown below. They are identical until the 39th place and therefore the boundary line between them is part of the level-39 (inverse) web but not the level-38 (inverse) web. On the magenta web this is true, but it is clearly false on the blue web.

Step sequence for p: $V_p[p,50] = 1, 1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 1, 3, 1, 2, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1, 3, 1, 2, 2, 1, \dots$

Step sequence for q: $V_p[q,50] = 1, 1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 1, 3, 1, 2, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1, 3, 1, 2, 2, \mathbf{2}, \dots$

Below is the inverse web to level 1200 in magenta and the corresponding forward web in blue. After 1200 iterations many differences still persist, but they are on a smaller scale so we have plotted them side by side below.



The blue forward web contains the same amount of detail, but it is detail about the inverse map σ . In any bounded region containing the origin, the web may 'stabilize' after a certain number of iterations and in this case the forward web and the inverse web will be identical and all subdomains will be periodic. This will occur with any lattice polygon.

References

F. Vivaldi, A. Shaidenko, *Global stability of a class of discontinuous dual billiards*.
Comm.Math. Phys. **110**, 625–640. (1987) MR895220 (89c:58067)