Introduction

This is an introduction to the $N = 2^k$ family of regular polygons. There is a brief analysis given for the first 5 members of this family in hopes of uncovering some non-trivial connection between them. It is not clear how these polygons are related dynamically, but geometrically they are closely related since each one is a 'bisection' of the previous. This means they form a sequence of nested factor graphs and it is no surprise that S[4] is 'mutated' for N = 16 and both S[4] and S[8] are mutated for N = 32. It would be useful to know how these mutations are related.

The algebraic complexity grows exponentially since $\text{EulerPhi}[2^k]/2 = 2^{k-2}$. Therefore N = 16 has 'quartic' complexity – along with N = 15 and N = 20. In [LKV] the authors note that algebraic analysis in the quartic case appears to involve "great computational difficulties".

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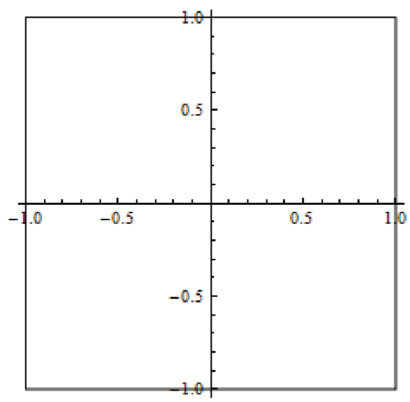
Summary of dynamics of the square: N = 4

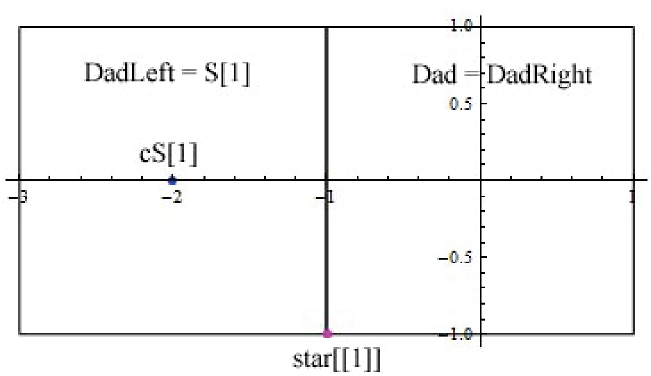
The square is the second regular lattice polygon - along with the equilateral triangle and the regular hexagon. Since our convention for 2k-gons is a height of 1, The tile for N = 4 will have vertices {1,1},{1,-1},{-1,-1}, {-1,1}. The 'M'-type tiles does not exist for N twice even so we will be dealing only with 'D'-type tiles, so N = 4 can also be called D.

Graphics[poly[D], Axes->True]



The First Family for N = 4 is shown below. It looks a little strange because we usually include DLeft and DRight (D) in the family. In the case of N = 4, these are the only 2 family members.

Graphics[{poly/@FirstFamily, Magenta,AbsolutePointSize[5.0],Point[star[[1]]], Blue,Point[cS[1]]},Axes->True]

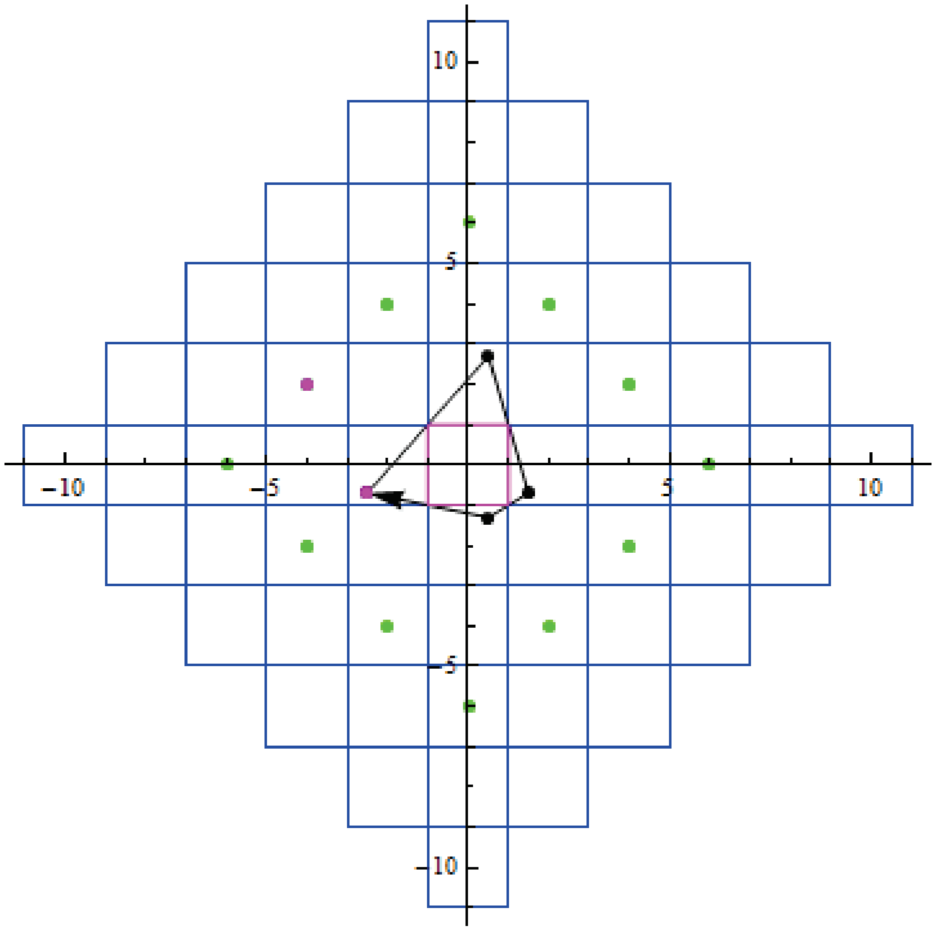


The magenta point is star[[1]], the only star point. The star points define centers, which in this case is cS[1] shown in blue. The centers then define family members, so DLeft is known as S[1] because he is step-1 relative to D. For any point in the interior of S[1], the orbit around D skips one corner on each iteration before returning home after 4 iterations. So the orbit is period 4 and there is a ring of 4 identical S[1]'s surrounding D. One such orbit is shown below overlaid on a web plot.

web[.01,10,100,0];WebPlot={};For[i=1,i<=npoints,i++,WebPlot = Union[WebPlot,Jxy[[i]]]];

q1= {-2.5,-.7}; K = V[q1,4]; (*First 4 points in the orbit of q1 It repeats after 4 points*)

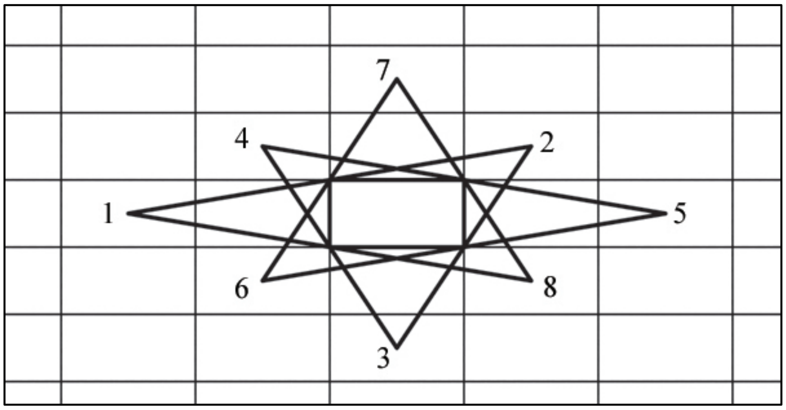
Graphics[{AbsolutePointSize[1.0], Arrow[K], AbsolutePointSize[5.0],Point[K] Blue,Point[WebPlot],Magenta,poly[D], AbsolutePointSize[5.0], Point[q1]},Axes->True]



The web is in blue and the initial point q1 is magenta. The second magenta dot is in ring 3 and we can see that it has period 12. The coordinates of this point are {-4,2} and since this is a lattice polygon every point in the orbit will have integer coordinates, for example $\tau(\{-4,2\}) = \{6,0\}$. This makes it easy to calculate orbits by hand with no round-off.

Ring k will have 4k Ds and this is also the period. In terms of step sequences, the first ring will have step sequence (1) and the points in the second ring will have step sequence (12) and each additional ring will add another 2 to the sequence, so the limiting sequence will be (2).

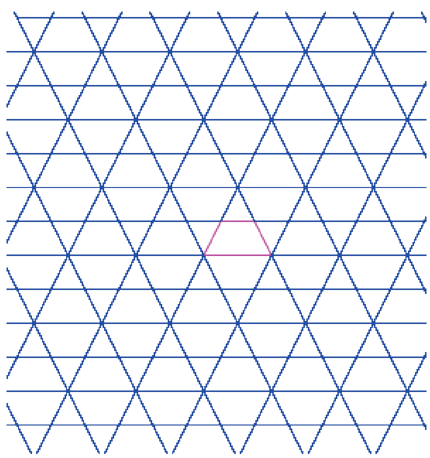
Any rectangle or affine transformation of a square, would have the same dynamics and periods. Shown below is a rectangle with width 2 and side 1/2. The orbit shown is the second ring of Ds so it is period 8.



For quadrilaterals which are not affinely equivalent to a square, such as the Penrose kite, there may be unbounded orbits, but D.Genin has shown that a trapezoid always has bounded orbits

2008: . D. Genin, *Research announcement: boundedness of orbits for trapezoidal outer billiards.* Electronic Research Announc. Math. Sci. **15** , 71–78. MR2457051 (2009k:37036)

The web for an arbitrary trapezoid is shown below. There is no affine transformation that would turn such a trapezoid into a kite.



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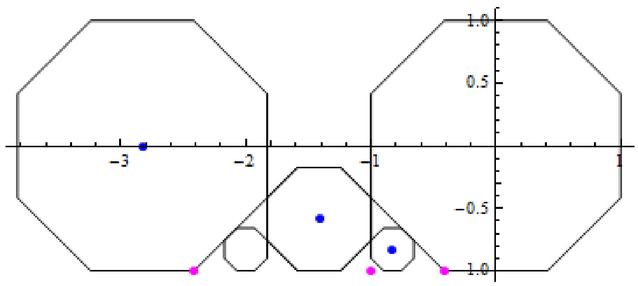
Summary of dynamics of the regular octagon: N = 8

N = 8 is the second 'twice-even' regular polygon. It is also part of the 'powers of 2' family of polygons. Unlike the 'twice-odd' families, there is typically no relationship between N = 2k and N = k, so the dynamics of N = 8 are largely unrelated to the dynamics of N = 4, and N = 16 bears little or no relationship with N = 8.

For regular N-gons with N even, our convention is to use a height of 1 instead of a radius of 1. We will also rotate them to orientate them 'upright'. This put vertex 1 at top right. In the notebook NTwiceEven.nb, the First Family for N = 8 is derived from the 'star' points as explained in N16Summary. That First Family is shown below along with the star points and centers. For the case of N even, we will usually refer to the generating polygon as 'Dad' instead of 'Mom'.

Graphics[{poly/@FirstFamily, AbsolutePointSize[5.0], Magenta, Point[{star[[1]],star[[2]],star[[3]]}], Blue, Point[{cS[1],cS[2],cS[3]}]}, Axes->True]



The horizontal displacements between the star points and the centers is always s0/2 where s0 is the side of Dad. This means that star points define centers and conversely. This is true for all regular polygons, because the star points define the bounds for the Tangent Map. In addition each star point defines a scale which is independent of the size of Dad.

scale[k_] := s0/(-2*star[k][[1]]); (*see the page on Scaling*)

Column[Table[scale[k], {k,1,HalfN-1}]]
1.00000000000000000000000000000
0.414213562373095048801688724210
0.171572875253809902396622551580

scale[3] ≈ 0.171572875253809902396622551580 is also known as **GenerationScale[8]** which we sometimes shorten to GenScale. The star point that corresponds to GenScale is known as GenStar. This is star[[3]] on the left above.

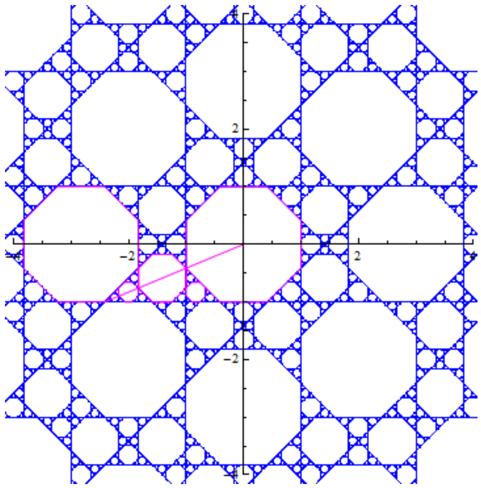
N = 8 has an infinite string of generations converging to GenStar and one of these is the remaining First Family member above. He is called LS[1] which is short for LeftS[1]. Note that LS[1] has the same size as S[1]. The large Dad on the left above is known as S[3] or simply DadLeft. His star points can be used to find the center of LS[1], or we can just rely on symmetry.

A web scan will uncover the basic dynamics of any regular polygon.

web[.01,5,300,0];WebPlot={};For[i=1,i<=npoints,i++,WebPlot = Union[WebPlot,Jxy[[i]]]];

box[{0,0},4]; (*crop region for the plot below*)

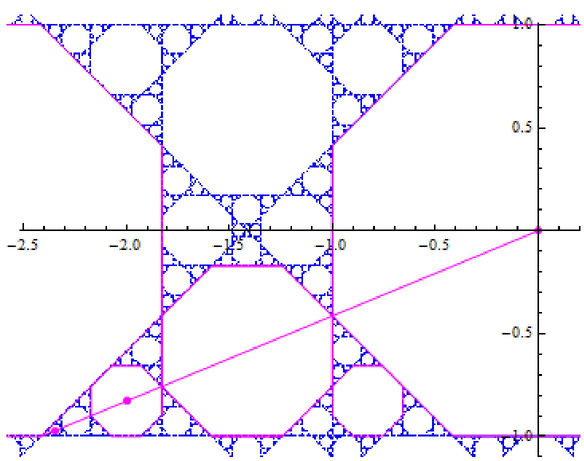
Show[Graphics[{AbsolutePointSize[1.0],Blue,Point[WebPlot],poly[Mom],Magenta,poly/@FirstFamily, Line[{0,0},GenStar}]},Axes->True,PlotRange->{{left,right},{bottom,top}}]]



The central region inside the ring of 8 Dads is invariant and serves as a template for the global dynamics. The line of symmetry shown above runs from the origin to GenStar. There is a chain of generations of Dads converging to GenStar and their centers all lie on this line. We can find these centers because GenerationScale[8] gives the scale for each generation relative to Dad, so the height of Dad[k] = (height of Dad)*GenScale^k = GenScale^k

Center from height = **CFR[r_]:= (1-h)*GenStar;** center of Dad[k] = CFR[Genscale^k].

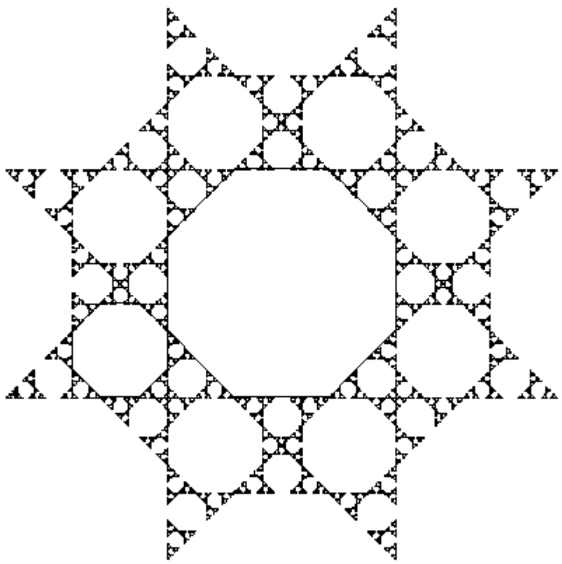
The first two centers in this chain are shown below (actually {0,0} is the first member).



The ratio of the periods of the Dads approaches 9: The first 7 terms in the sequence are of periods are 16, 96, 1008, 8640, 79056, 707616, 6382208,.. By the time we reach Dad[7] the orbits are virtually non-periodic and their density is very uniform. Below is the first 50,000 points in the orbit of Dad[7]:

cDad[7] = CFR[GenScale^7] ≈ {-2.4142029962396670,-0.9999956233642322980185191}

Orbit = V[cDad[7],50000];Graphics[{AbsolutePointSize[1.0],Point[Orbit]}] (*about 20 seconds*)



This implies that the fractal dimension of the 'web' is Ln[9]/Ln[1/GenScale] ≈ 1.24648 which is close to N = 5 at Ln[6]/Ln[1/GenScale[5]] ≈ 1.24114. They are both sparse compared to the classic Sierpinski triangle at Ln[3]/Ln[2] ≈1.58496.

All the regular N-gons of the form $N = 2^k$ are constructible because they can be constructed from the square by successive angle bisections. If N is one of these ‘powers of 2’ regular polygons, then φ(N) (EulerPhi) = N/2 and the degree of the minimal polynomial for Cos[2Pi/N] is always half of this.

For N = 8, φ(8) = 4 and **MinimalPolynomial[Cos[2*Pi/8]][x]** = $2x^2-1$ so Cos[Pi/4] = $1/\sqrt{2}$

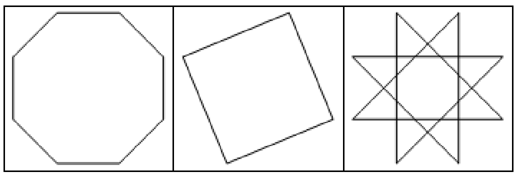
This implies that the vertex set lies in $\mathbb{Z}(\sqrt{2})$ in a fashion similar to the regular pentagon.It is no coincidence that the webs for both of these are self-similar fractals.

Projections

N = 8 has only one non-trivial projection but our convention will be to show all the possible projections. In this case the 'redundant' P2 projection is also interesting. The projections clearly show the self-similar nature of the orbits.Each orbit is a 'refinement' of the previous.

The three possible mappings of the vertices are stored in the matrices Wc[[1]], Wc[[2]] and Wc[[3]]]. These are shown below

GraphicsGrid[{Table[Graphics[poly[Wc[[k]]]],{k,1,HalfN-1}]},Frame->All]



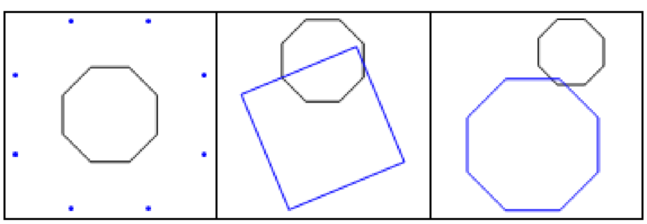
Below we will show the projections for the sequence of Dads

Example 1: cDad[1] = CFR[GenScale] ≈{-2.0,-0.82842712474619009760 3}
This point is period 16, so the projections will have period 8

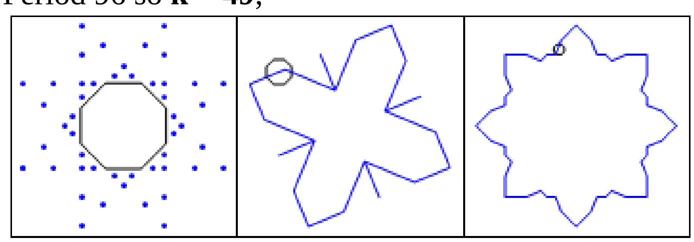
Ind = IND[q1,20]; (*generate first 20 vertices in the orbit*)
k = 9; (*Plot one more than half the period to close the plot*)

Px=Table[Graphics[{poly[Mom],Blue,Line[PIM[q1,k,j]]}],{j,1,HalfN-1}];
(*generate the three projection using pairs of vertices from Ind*)
Px[[1]]=Graphics[{poly[Mom],AbsolutePointSize[3.0],Blue,Point[PIM[q1,k,1]]}];
(*plot the P1 projection with points instead of lines*)

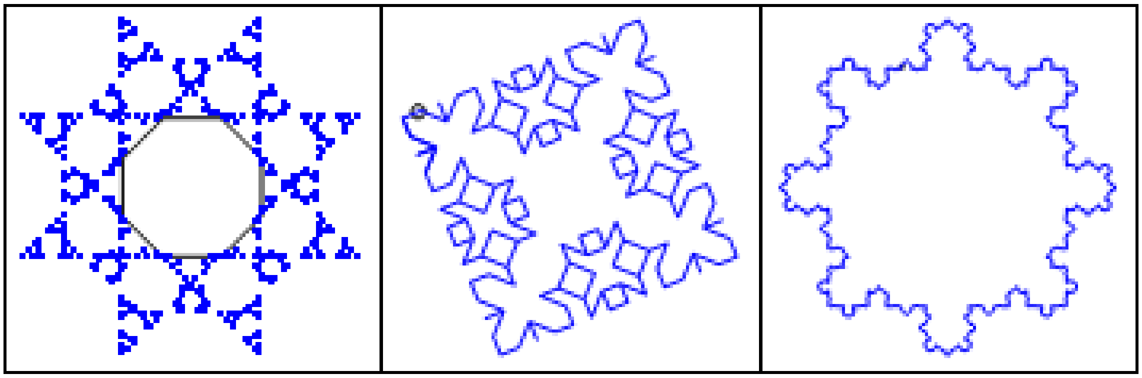
GraphicsGrid[{{Px[[1]],Px[[2]],Px[[3]]}},Frame->All]



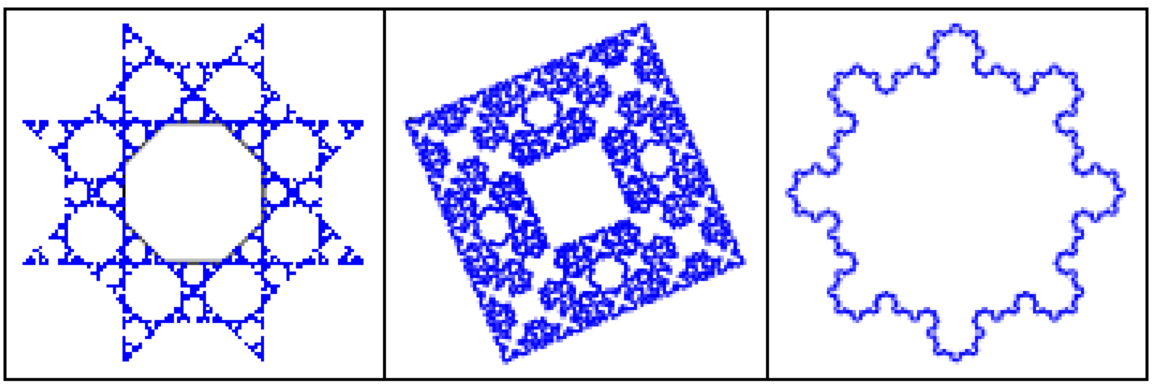
Example 2: cDad[2] = CFR[GenScale^2] ≈ {-2.34314575050761,-0.970562748477140585}
Period 96 so **k = 49**;



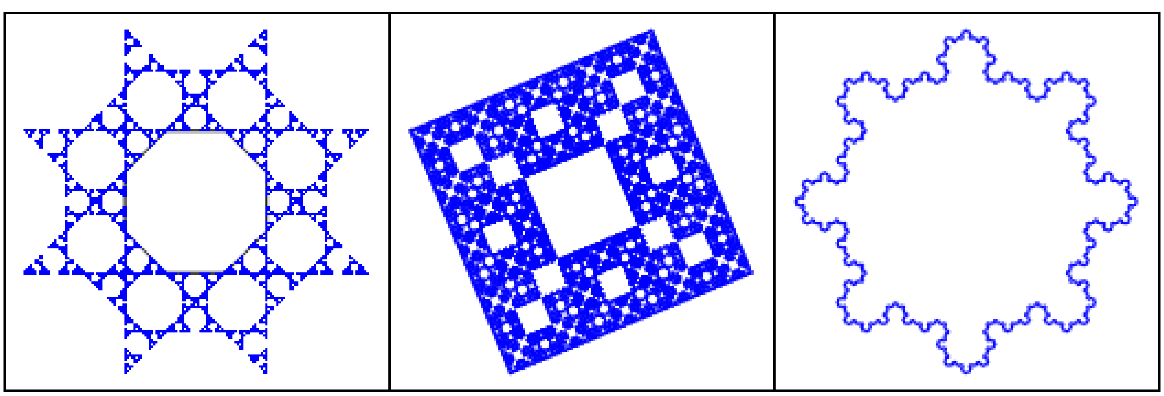
Example 3: cDad[3] = CFR[GenScale^3] ≈ {-2.4020202535533,-0.994949366116653416}
Period 1008 so **k = 505;**



Example 4: cDad[4] = CFR[GenScale^4] ≈{-2.41212152132003,-0.99913344822277991108}
Period 8640 so **k = 4321;**



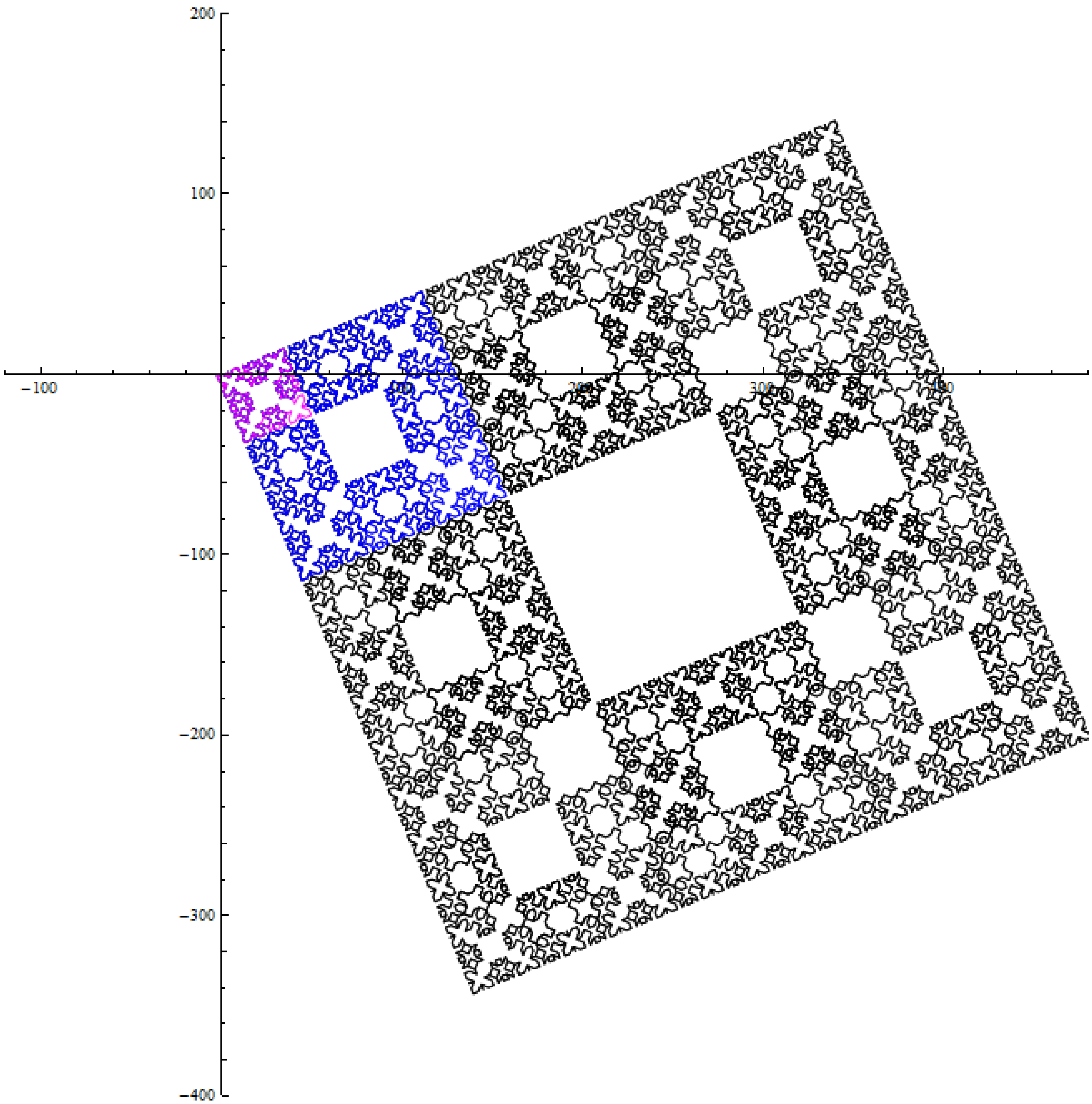
Example 5: cDad[5] = CFR[GenScale^5] ≈ {-2.413854624874471,-0.99985132322002605 05}
Period 79056 so **k = 39529;**



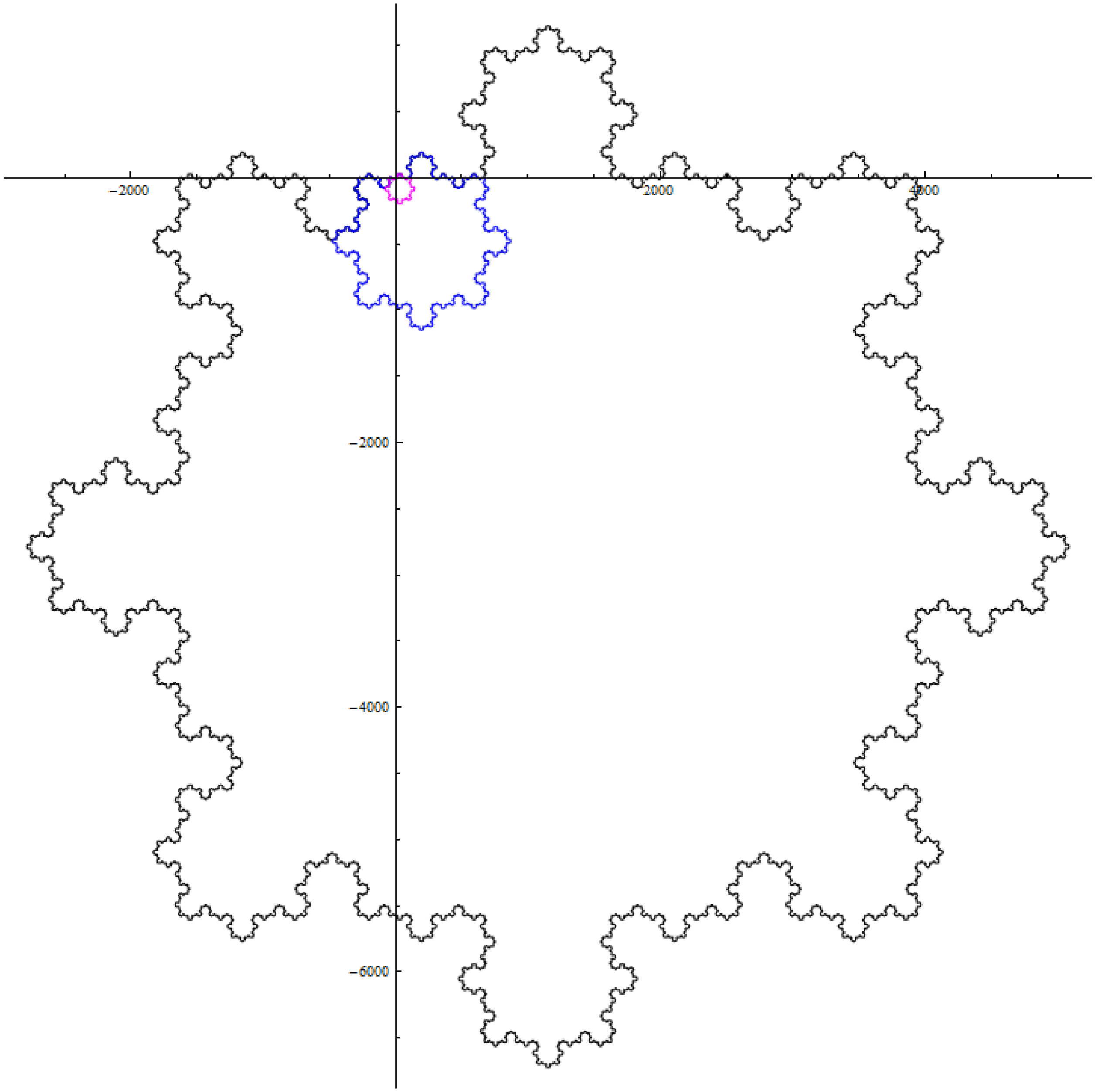
These projections are increasing in scale and they are 'nested' . To see this we will plot the last three in different colors below using Dad[3] (magenta) ,Dad[4] (blue) and Dad[5] (black):

Ind= IND[cDad[3],2000]; Px32= PIM[cDad[3],520,2]; Px33=PIM[cDad[3],520,3];
Ind= IND[cDad[4],10000]; Px42= PIM[cDad[4],4500,2]; Px43=PIM[cDad[3],4500,3];
Ind= IND[cDad[5],100000]; Px52= PIM[cDad[5],40000,2]; Px53=PIM[cDad[3],40000,3];

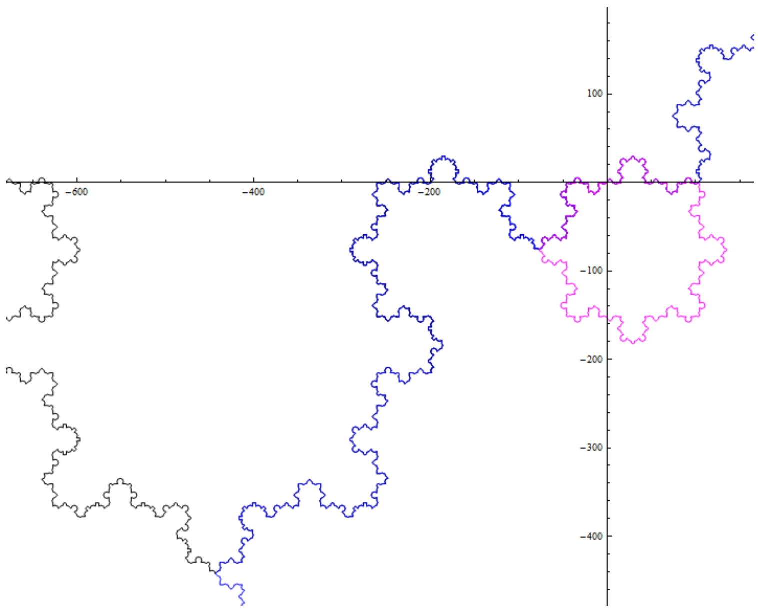
Graphics[{AbsolutePointSize[1.0],Line[Px52], Blue, Line[Px42],Magenta,Line[Px32]},Axes->True]



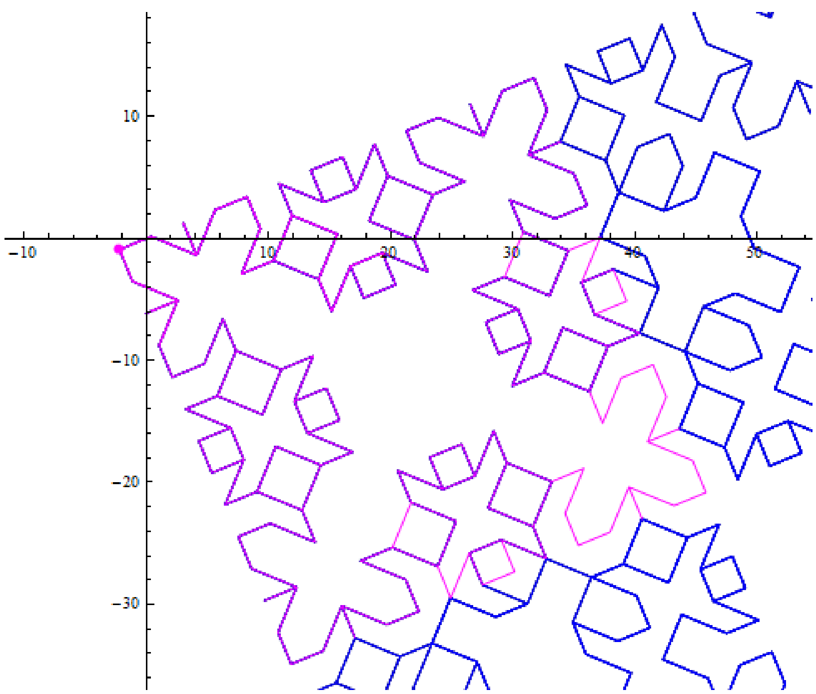
Graphics[{AbsolutePointSize[1.0],Line[Px53], Blue, Line[Px43],Magenta,Line[Px33]},Axes->True]



This chain is clearly endless. Since the periods grow by a factor which approaches 9 we can renormalize this chain to turn this large-scale fractal structure into a Koch snowflake whose fractal dimension should be Ln[9]/Ln[1/GenScale] ≈ 1.24648 . The motion is clockwise. Below is an enlargement of the region around Mom. Note that the Blue and Black branch points are self-similar



Below is a similar enlargement of the 'sponge'. In the limit it should have the same fractal dimension as the snowflake above. cDad[3] is shown as a Magenta dot. The dynamics here are complicated by the redundancy so the branching is highly non-trivial.



Richard Schwartz at Brown University obtained similar plots using Pinwheel projections. The projections shown above are not Pinwheel projections but they are a type of Arithmetic Graph that has proven useful in his quest to track unbounded orbits. For more information see the section on Projections, Pinwheel Maps and Arithmetic Graphs.

References

Schwartz R.E. , Unbounded Orbits for Outer Billiards, Journal of Modern Dynamics 3 (2007)

Schwartz R.E. , Outer Billiards on Kites, Annals of Mathematics Studies, 171 (2009), Princeton University Press, ISBN 978-0-691-14249-4

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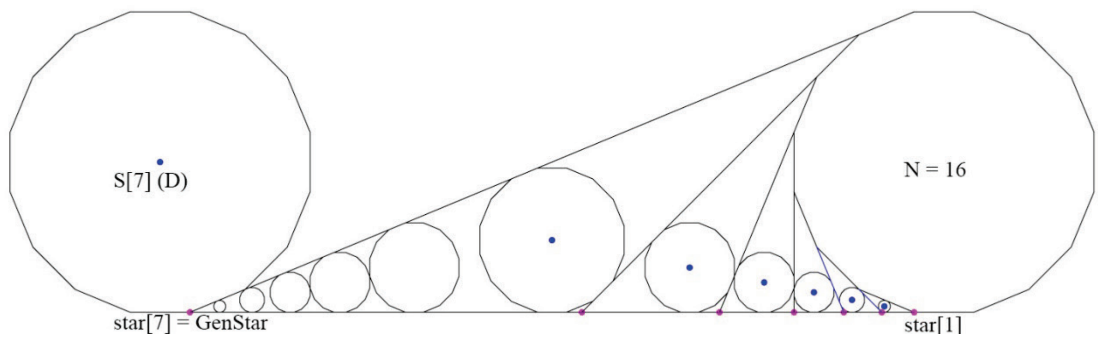
Summary of dynamics of the regular hexadecagon: N = 16

N = 16 is the first non-trivial member of the N = 2^k family. It is not clear how the members of this family are related dynamically, but geometrically they are closely related since each one is a 'bisection' of the previous. This means they form a sequence of nested factor graphs and it is no surprise that S[4] is 'mutated' for N = 16 and both S[4] and S[8] are mutated for N = 32.

The algebraic complexity grows exponentially since EulerPhi[2^k]/2 = 2^{k-2}. Therefore N = 16 has 'quartic' complexity – along with N = 15 and N = 20. In [LKV] the authors note that algebraic analysis in the quartic case appears to involve "great computational difficulties".



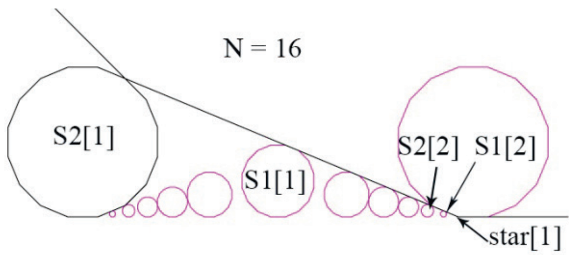
Below are the periods of the First Family where S[6] = LS[6] occupies the central position. Note that it is period 8 because of decomposition. S[2], S[4] and S[6] also have shortened periods, but S[4] is the only 'mutated' tile. It consists of two interwoven squares at slightly different radii.

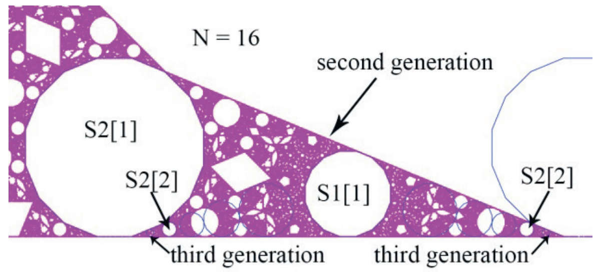
Tile	S[1]	S[2]	S[3]	S[4]	S[5]	S[6]	LS[1]	LS[2]	LS[3]	LS[4]	LS[5]	LS[6]
Period	16	8	16	4	16	8	96	40	64	12	32	8

Using the 2kN Lemma of the Digital Filter map, these orbits can be united as follows. Each S[k] and LS[k] form a combined count which can be found using the Df periods of the S[k]. This is shown below.

Tiles	S[1]& LS[1]	S[2]& LS[2]	S[3]& LS[3]	S[4]& LS[4]	S[5]& LS[5]	S[6]& LS[6]
Count	2*7*8	2*6*8	2*5*8	2*4*8	2*3*8	2*8

In twice-even cases. the canonical tiles are all N-gons – so it appears that the M-D distinction does not exist. However the notion of 'generations' for N-odd or twice-odd, still exists. When N is odd, each generation (if it exists) is presided over by an M[k],D[k] pair playing the roles of matriarch and patriarch. These canonical M[k] and D[k] tiles form on the edges and vertices of the previous D[k-1], so they are always step-1 and step-2 respectively. N = 16 preserves this step-2 vs. step-1 dichotomy, so it would not be 'politicially incorrect' to associate S[1] with M[1] and S[2] with D[1]. These two tiles are shown on the left along with a magenta virtual First Family for S2[1].

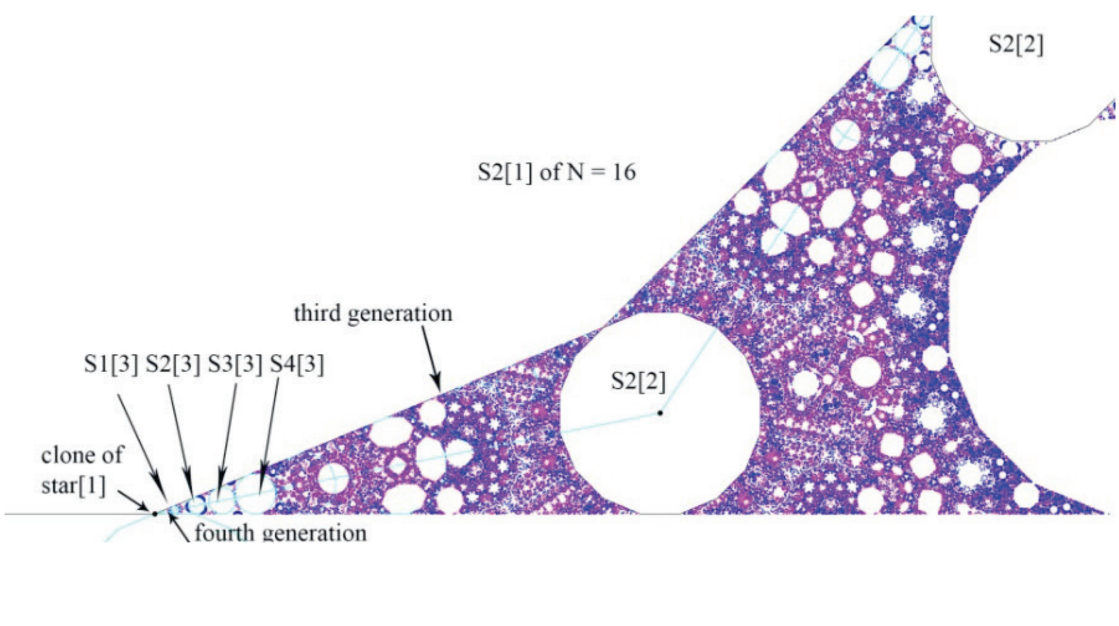




Note that S1[1] is a step-6 of S2[1] so it is natural to associate this tile with M[1]. However the web plot on the right makes it clear that S1[2] does not exist, so there is no 'M[2]]'. Since S2[2] does exist at star[1] and at the foot of S2[1], either tile can play the role of D[2]. We choose to study the foot of S2[1] because it provides valuable information about the local dynamics – which is often 'hidden' at star[1] and at GenStar.

This region is enlarged below – note that the symmetry now is with respect to S2[2]. We have reproduced the virtual First Families of S2[2] to show the perfect match with four members of the third generation- including S1[3] which is the surrogate M[3]. The first 10 S2[k]'s in this sequence have periods 8, 32, 456, 2464, 20872, 110368, 974664, 5165216, 45423368 and 240668192 which gives ratios of about 4.66 and 8.8.

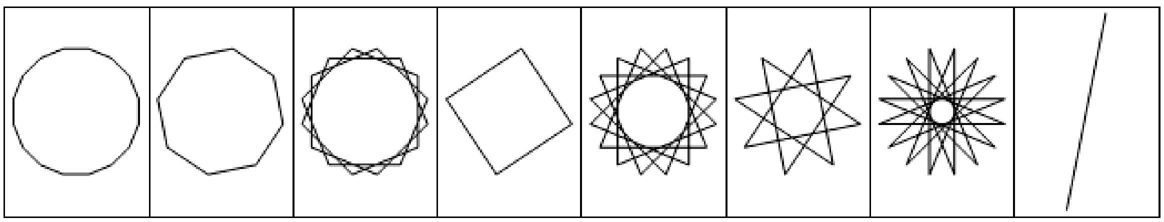
As with N = 7, the ratios alternate high-low within the even and odd sequences. N=20 may also support generations – but they are complicated by the fact that S2[1] is a mutated decagon.



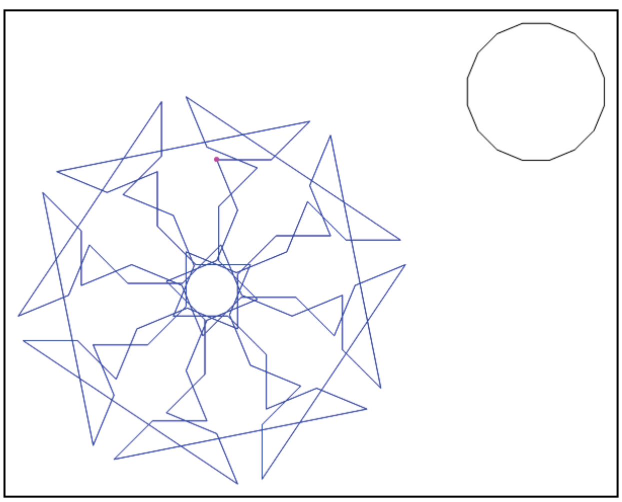
Projections

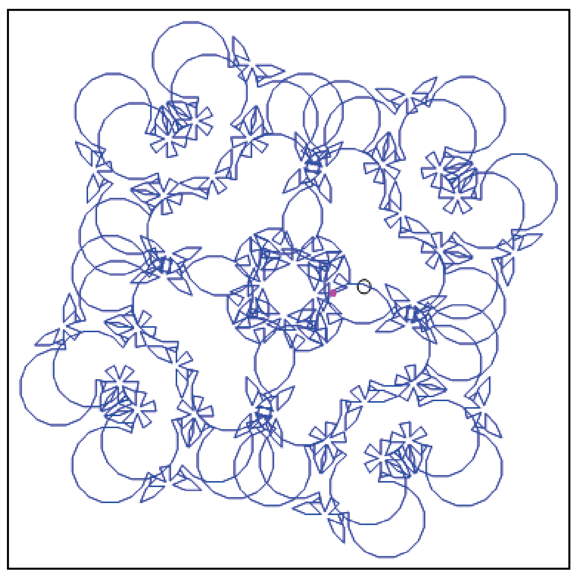
EulerPhi[16] = 8 so there are 4 non-redundant projections but our convention is to look at all 7 . Below are the vertex assignments for the projections, showing 3, 5 and 7 as the relatively prime projections.

GraphicsGrid[{ Table[Graphics[poly[Wc[[k]]]],{k,1,8}]},Frame->All]



Example 1: The magenta dot in D[2] below has period 176 and the second dot, where M[2] should be, has period 2952. The projections have periods which are half of these. Below is the P7 projection for D[2] and the P5 projection for 'M[2]'.



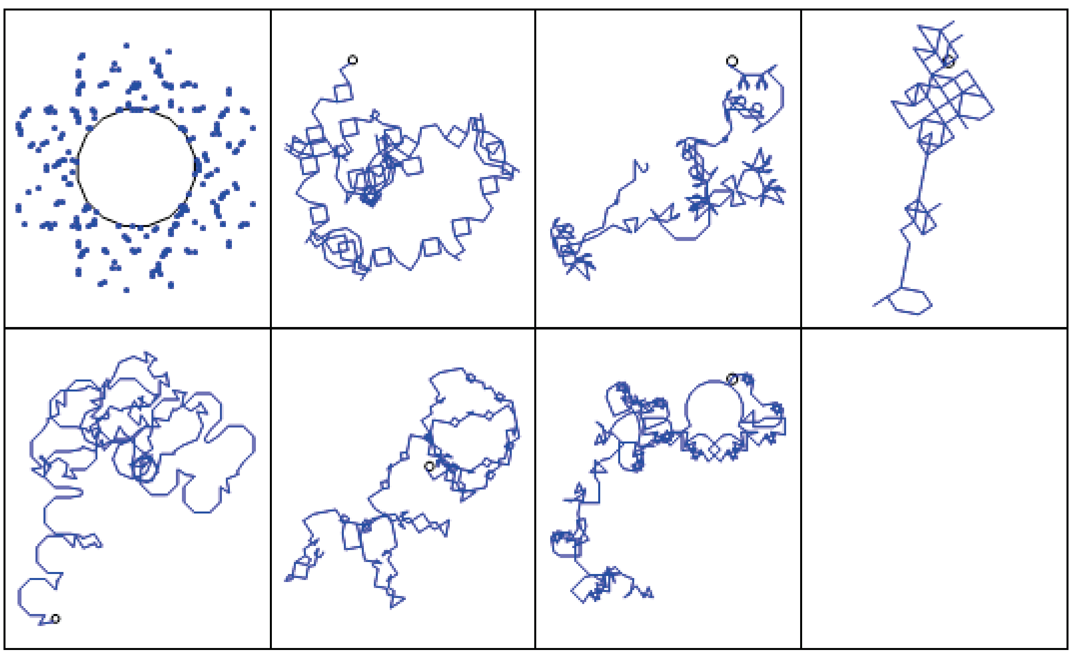


The following are samples of projections from each of the three invariant regions:

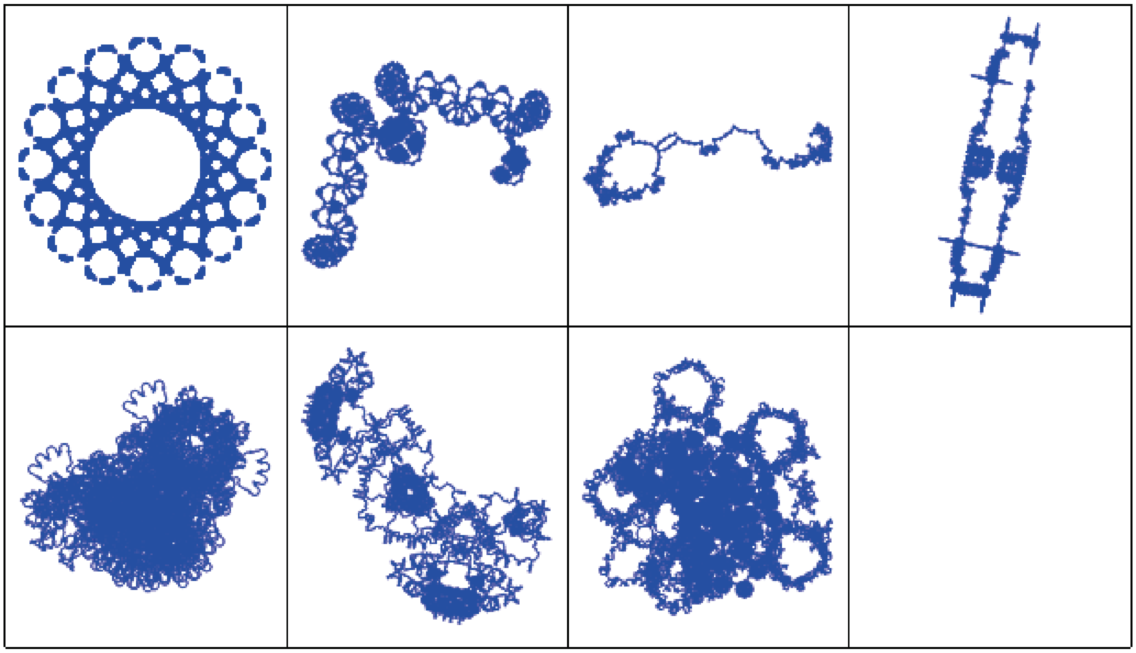
Example 2: The Inner ring: q1= {-.67013678100000001233, -.75300001627788811};

Ind = IND[q1, 50000]; k= 250; (*plot the first 250 iterations*)

GraphicsGrid[{{Graphics[{poly[Mom],Blue,Point[PIM[q1,k,1]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,2]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,3]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,4]]}]},
{Graphics[{poly[M],Blue,Line[PIM[q1,k,5]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,6]]}],
Graphics[{poly[M],Blue,Line[PIM[q1,k,7]]}]}},Frame->All]

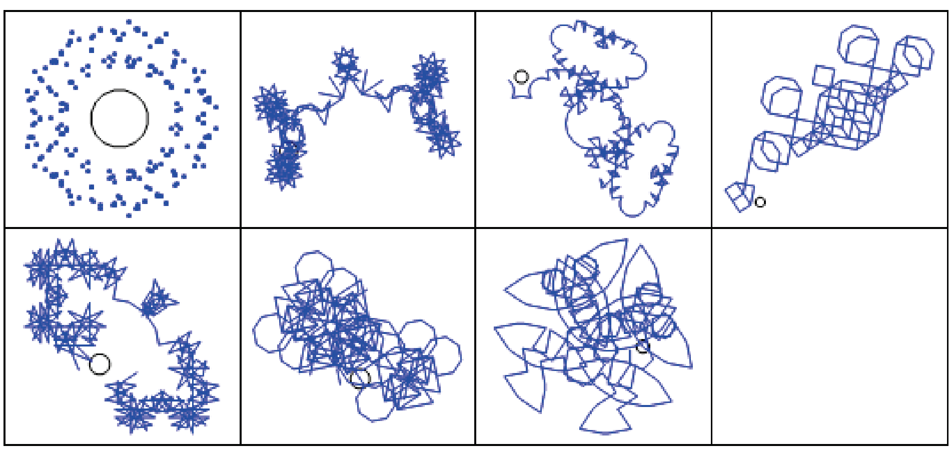


Now **k = 20000**

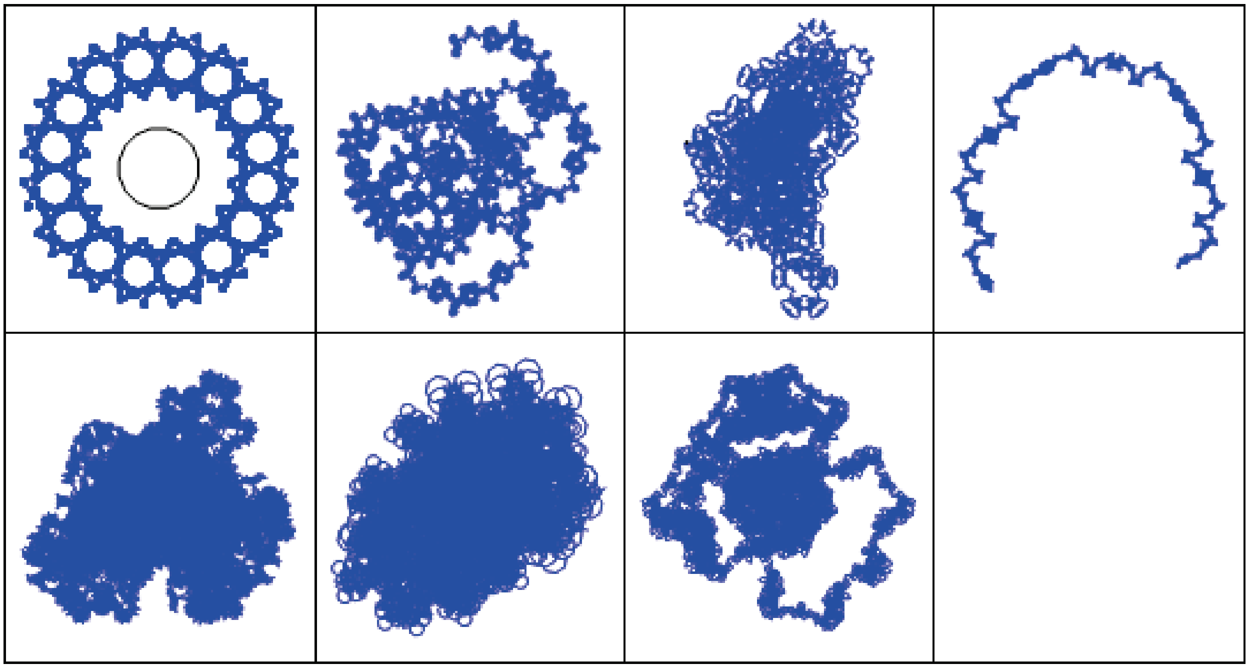


Example 3: The middle ring: q1= {-2.08112357199000010110, -.53262317191122200001}

k = 250

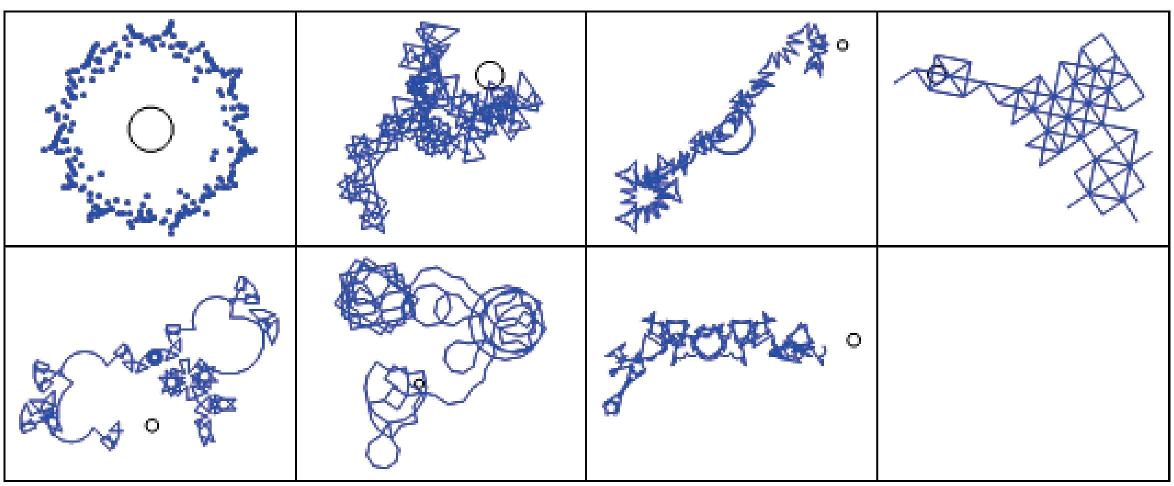


k = 20000. Note the increase in density as we move outwards.

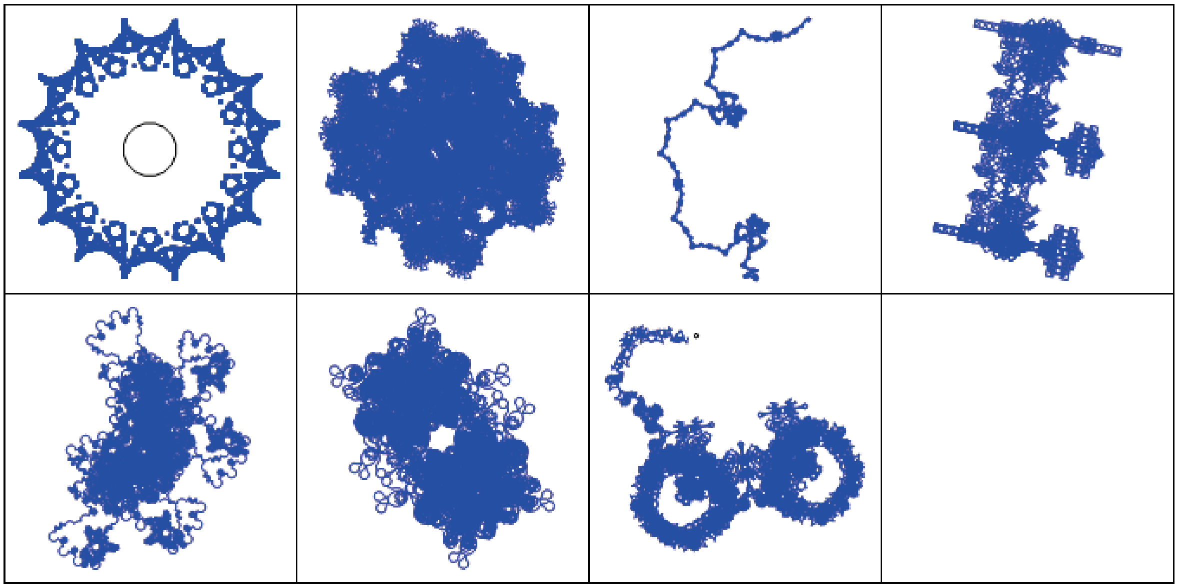


Example 4: The outer ring: q1= {-4.9012667100111113, -.973701771231000001}

k = 250;

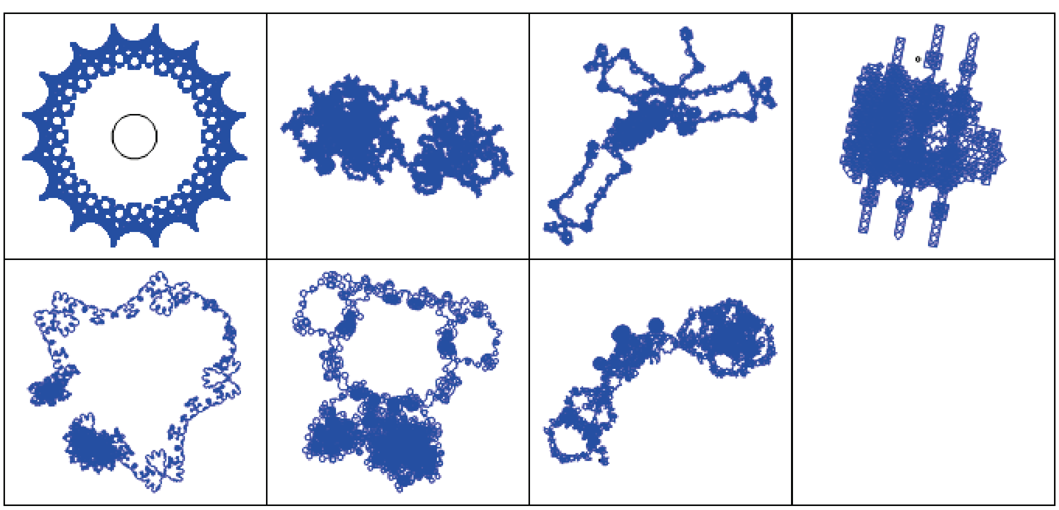


k = 20000. Yes, P7 is a bicycle and P5 is waving frantically.

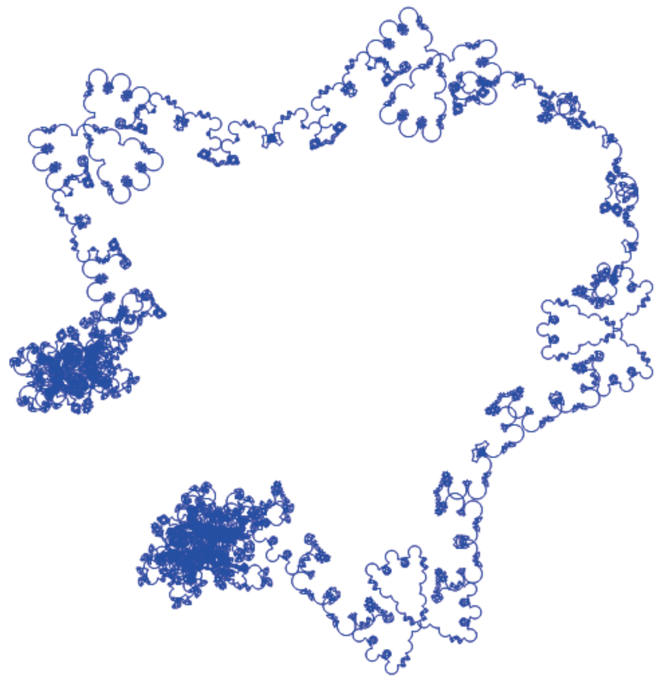


This last point has been a 'work-horse' because it is remarkably dense in generation 2 and beyond. (See the earlier plots.) Since we have tracked the orbit for billions of iterations we can skip ahead in the projections using an advanced initial point.

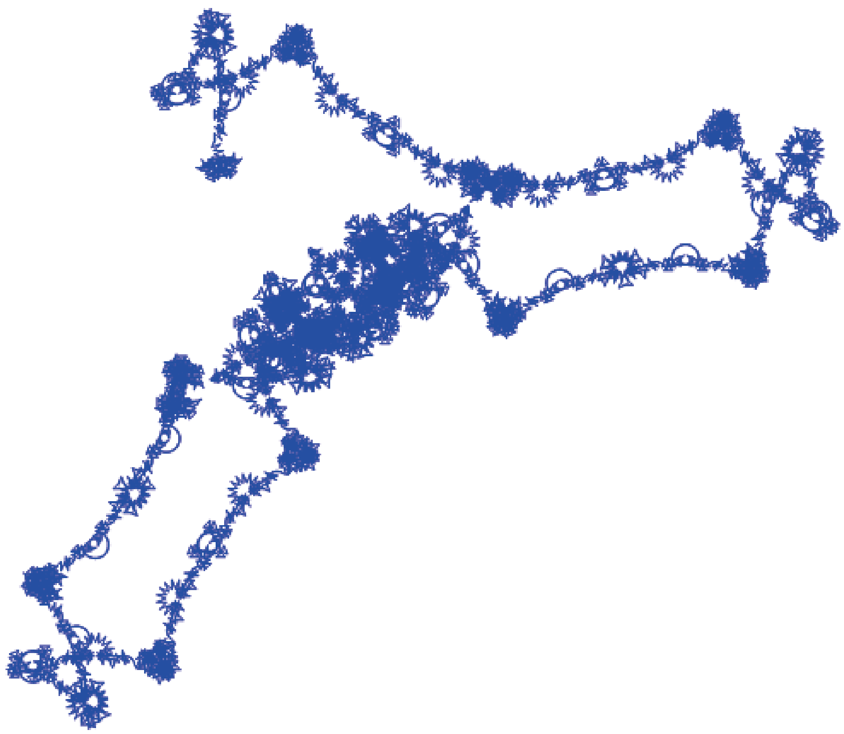
Example 5: q1 = {3.6910131180033360897388,0.0929005697712447137357} which is an advance of 65 million iterations from q1 in Example 4 above.
k = 20000



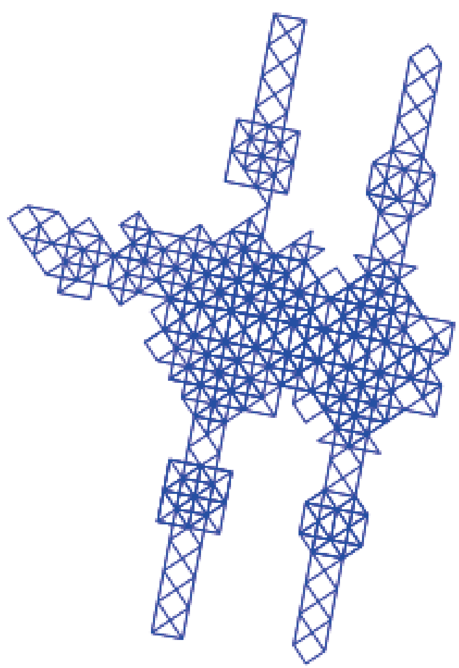
Below is an enlargement of P5 showing just the first 15,000 points. The hands which had four fingers now have three.**Graphics[{Blue,Line[PIM[q1,15000,5]]}]**



All the projections share the same symmetry. Below is P3 for these same parameters.
Graphics[{Blue, Line[PIM[q1, 15000, 3]]}]



As expected, P4 is an 'erector set'. Below are the first 4000 points.
Graphics[{Blue,Line[PIM[q1,4000,4]]}]



References

Hughes G.H., Outer billiards on regular polygons, **arXiv:1206.5223**

[LKV]Lowenstein J. H., Kouptsov K. L. and Vivaldi . F, Recursive tiling and geometry of piecewise rotations by π/7, Nonlinearity 17 1–25 MR2039048 (2005)f:37182)

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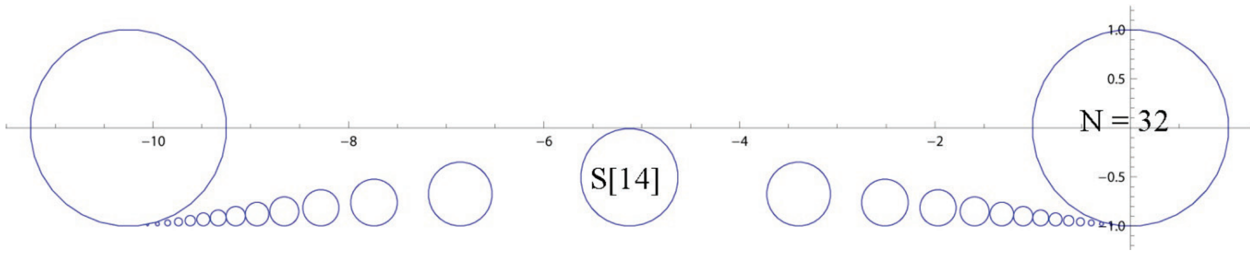
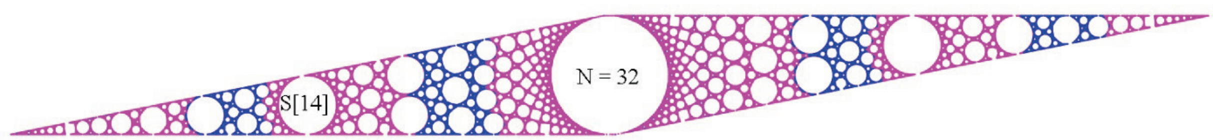
Summary of dynamics of N = 32

N = 32 is the fourth member of the 2^k family. N = 4 has a trivial web which consists of just the extended trailing edges, N = 8 has a self-similar fractal web, N = 16 is probably multi-fractal but it does appear to support infinite families of S[1] and S[2] tiles which play the role of M[k] and D[k] – except that the M[k]'s are only present on the odd generations. This means there is hope that N = 32 may also have such families – but we have found no evidence of this.

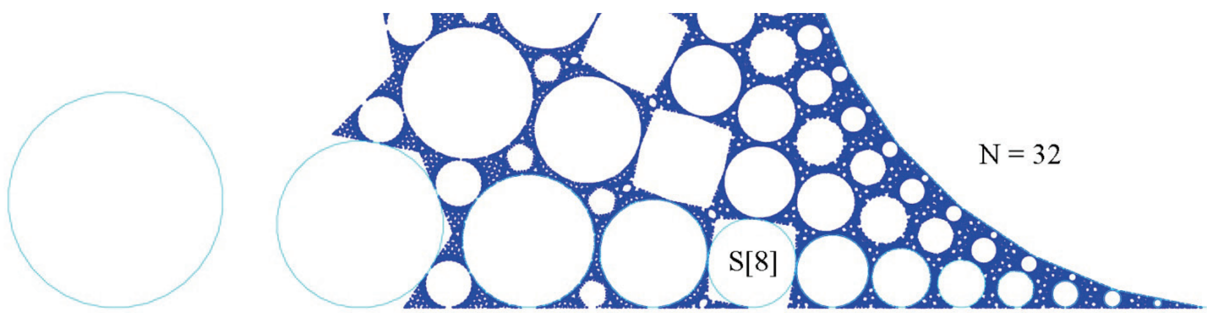
Algebraically N = 32 is degree 8 since EulerPhi[32] = 16. The canonical First Family is shown below – but there are the expected mutations in S[4], S[8] and S[12] – which are best viewed using a web plot.



The five invariant regions are shown below in a toral Digital Filter plot.

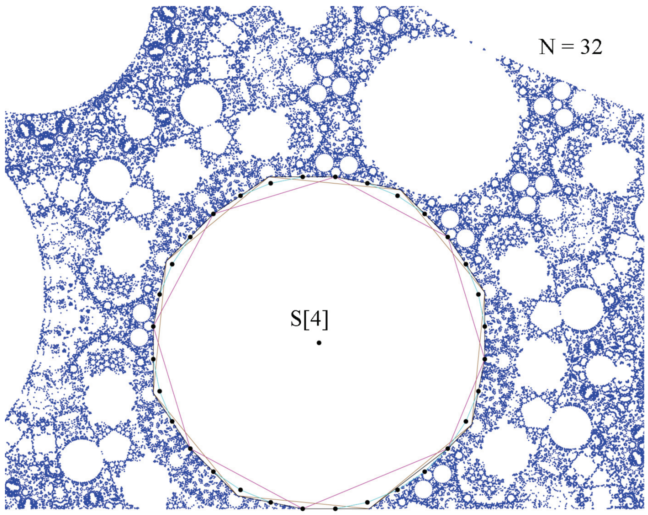


The invariant inner family is shown below in Blue – so that that the reader is not 'blinded' by the magenta from the plot above. As expected, S[4] and S[8] are mutated tiles based on the canonical 'templates'. S[8] is a more extreme mutation than S[4] – which looks almost normal at this scale.

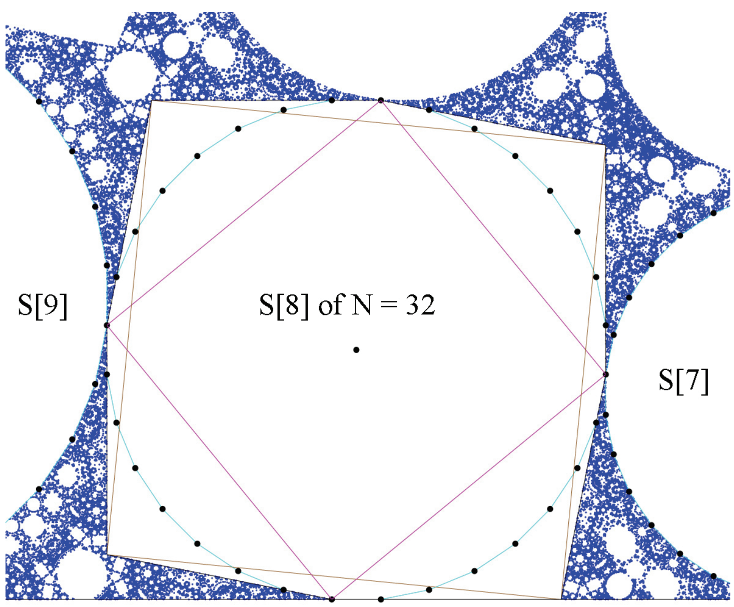


Mutations in S[4] and S[8]

The S[4] orbit 'sees' only a subset of the N = 4 and N = 8 polygons embedded in N = 32, so it evolves in a 'mod-4' fashion consisting of two interwoven regular 16-gons at slightly different radii. These are the magenta and brown 16-gons shown below. This is the same type of mutation found with S[3] of N = 9. The canonical S[4] 'template' is shown in Cyan



The step-8 orbit also 'sees' a subset of the edges of N = 32 and this partitions the vertices 'mod-8'. Therefore the resulting tile consists of two perfect squares which are shown in magenta and brown below. Once again the mutated tile shares the same center as the canonical regular tile.

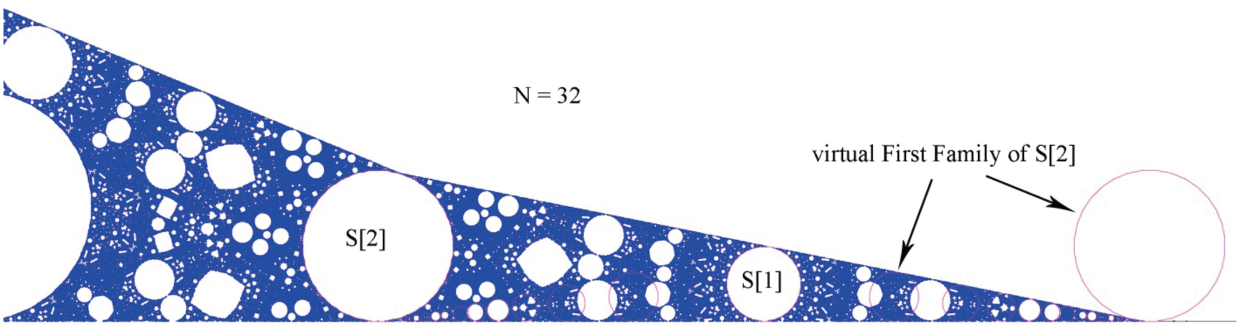


These mutations always have matching decomposition of orbits – so the periods of S[4] and S[8] are 8 and 4 respectively. It is not clear how these mutations affect subsequent generations.

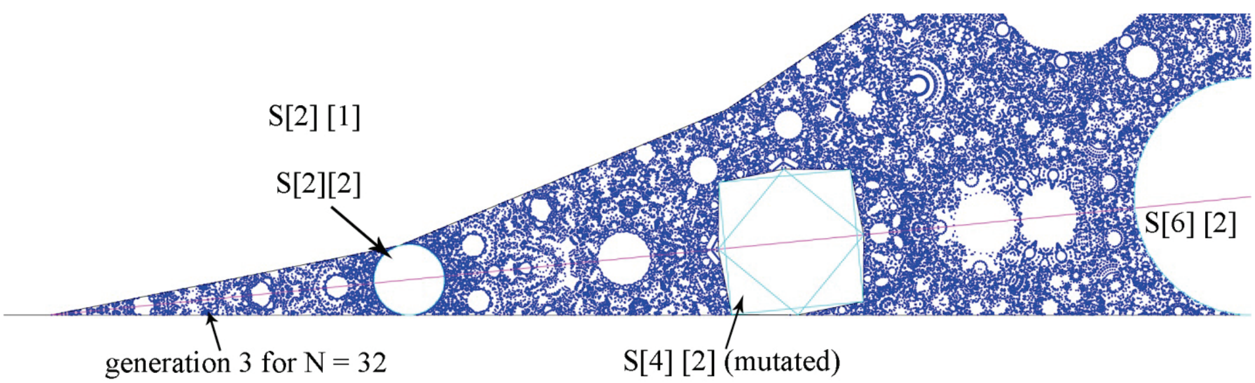
The Second Generation

Below is the second generation where S[2] and S[1] have the potential to play the role of D[1] and M[1] and foster future generations at the foot of S[2] or at star[1] of N = 32 – which is D[0]. Clearly there is symmetry between these two points. The virtual First Family shown here is a scaled version of the N = 32 First Family. The Mathematica commands for scaling and importing this virtual First Family are shown below:

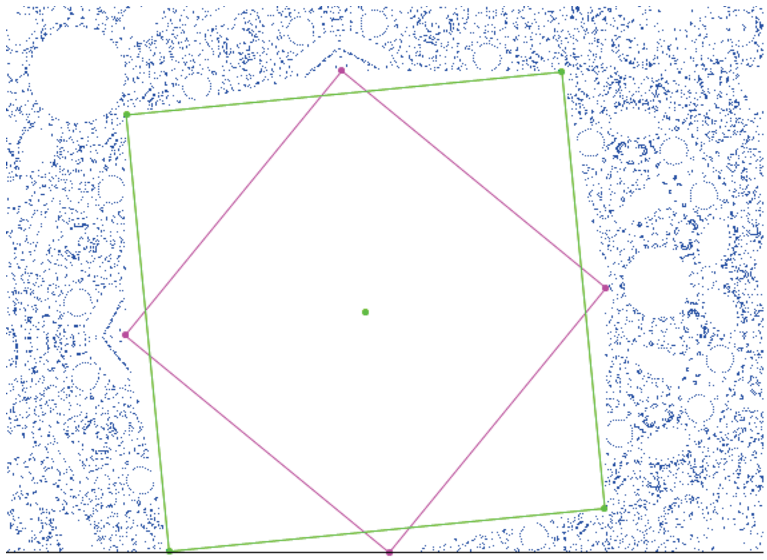
FFS2 = TranslationTransform[cS[2]] /@ (FirstFamily*GenScale/scale[2]);
FFS2 = ReflectionTransform[{1, 0}, cS[2]] /@ FFS2;



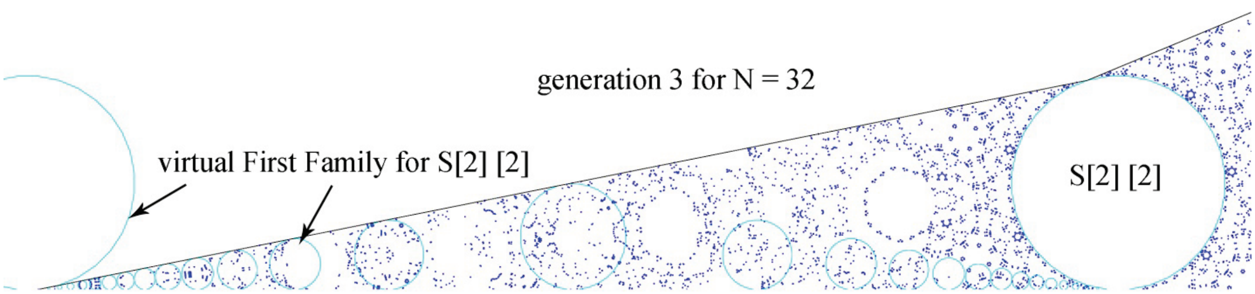
Note that S[1] plays the role of the central S[14] from the original First Family. This is promising, but the real issue is whether there is any continuation of this structure on the edges of S[2]. This region is enlarged below. S[2][2] and S[6][2] are the only surviving canonical tiles and S[4][2]] is mutated in a fashion identical to S[8] from the first generation.



The green and magenta squares shown below have radii approximately 0.00102674 and 0.00079751 and these are scaled by GenScale[32]/scale[2] from the radii in S[8], so the ratio is both cases is ≈ 0.776745.



The third generation shown below has no S[1] [2] but this was true for N = 16 also and the S[1] tiles returned on the next generation. This does not occur here. There is no indication of further family structure around S[2][2] or at the local GenStar point, but there may be self-similar families elsewhere.



The dynamics local to S[14]

When N is twice even, the possible step sizes for Df webs runs from 1 to N/4 -1, so for N = 32 there are Df webs with rotation numbers of 1/32 to 7/32– some of which reduce to known cases.

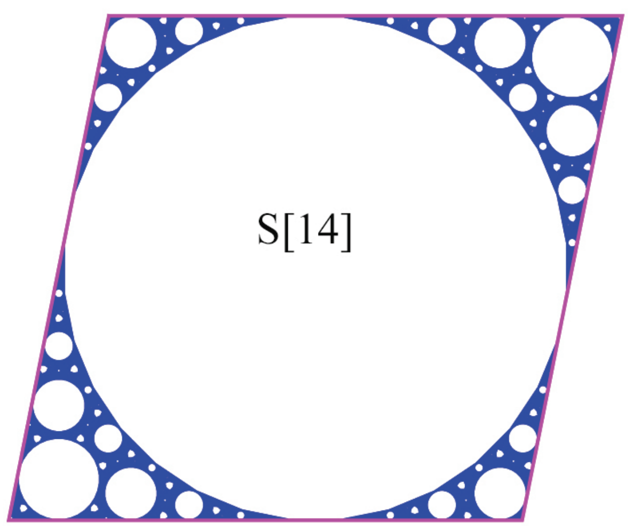
The 'maximal' step-7 web is very useful because it reproduces the local dynamics of the S[14] tile. Since the 'twist' ρ = 7/32 , the Df map θ is 2π(7/32) = 7π/16 as shown below:

```
DigitalFilter = Module[{}, w= N[7*Pi/16];a=2*Cos[w]; f[x_]:=Mod[x+1,2]-1;
Df[{x_,y_}]:={y,f[-x+a*y]};
Shear[{x_,y_}]:={x,x *Cos[w]+y *Sin[w]}; IShear[{x_,y_}]:={x,-x *Cot[w]+(y /Sin[w])};
DfToTr[A_]:=If[Length[A]==2,RotationTransform[Pi/2][IShear[A]],RotationTransform[P
i/2][IShear/@A]];
TrToDf[A_]:=If[Length[A]==2,Shear[RotationTransform[-
Pi/2][A]],Shear/@RotationTransform[-Pi/2][A]];
```

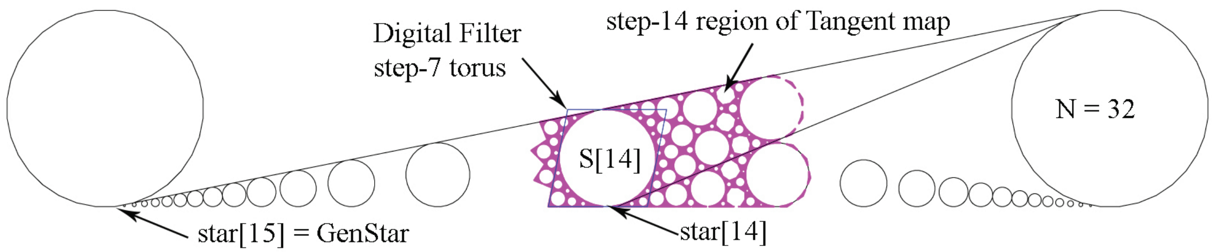
The web plot below shows some of the 'First Family' for S[14]. The only match that exists with the First Family for N =32 is S[2]. This tile is the same size as the S[2] of N = 32 because by convention, S[14] is 'promoted' in scale to match N = 32. This does not mean that S[14] has dynamics similar to N = 32 because a step-7 web is very different than a step-1 web. Aside from S[2], the large step tiles shown below for S[14] are not canonical in position or size – some are too large and some are too small compared with N = 32.

Below is a web scan of depth 1000, One possible choice for initial segment H0 is the top edge of the Df torus which runs from {-1,1} to {1,1}, but at this scale only a short sub-section is needed.

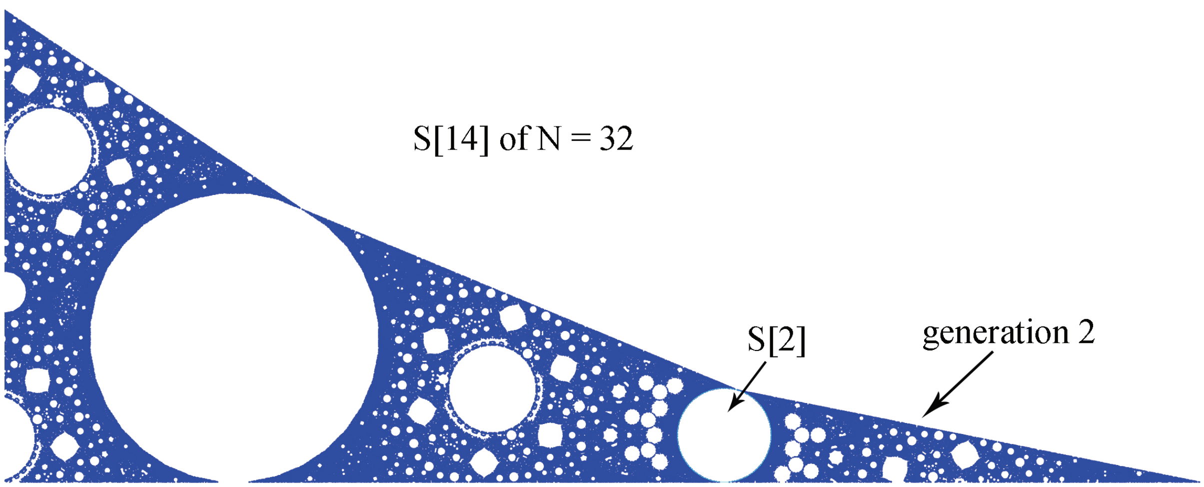
```
H0 = Table[{x, 1}, {x, .9, 1, .005}];
DfWeb =  Flatten[Table[NestList[Df, H0[[k]], 1000], {k, 1, Length[H0]}], 1];
W1 = DfToTr[DfWeb]; Graphics[{Blue, AbsolutePointSize[1.0], Point[W1]}]
```



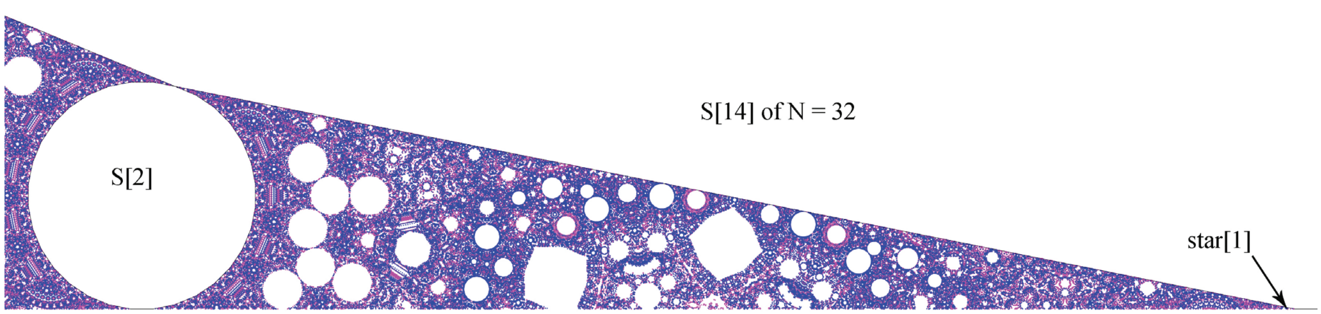
This Df step-7 web replicates a portion of the step-14 Tangent map web as shown below. The magenta region around S[14] is invariant and the blue Df torus matches the left-side bounds of the invariant region – so the Df web replicates the dynamics local to S[14].



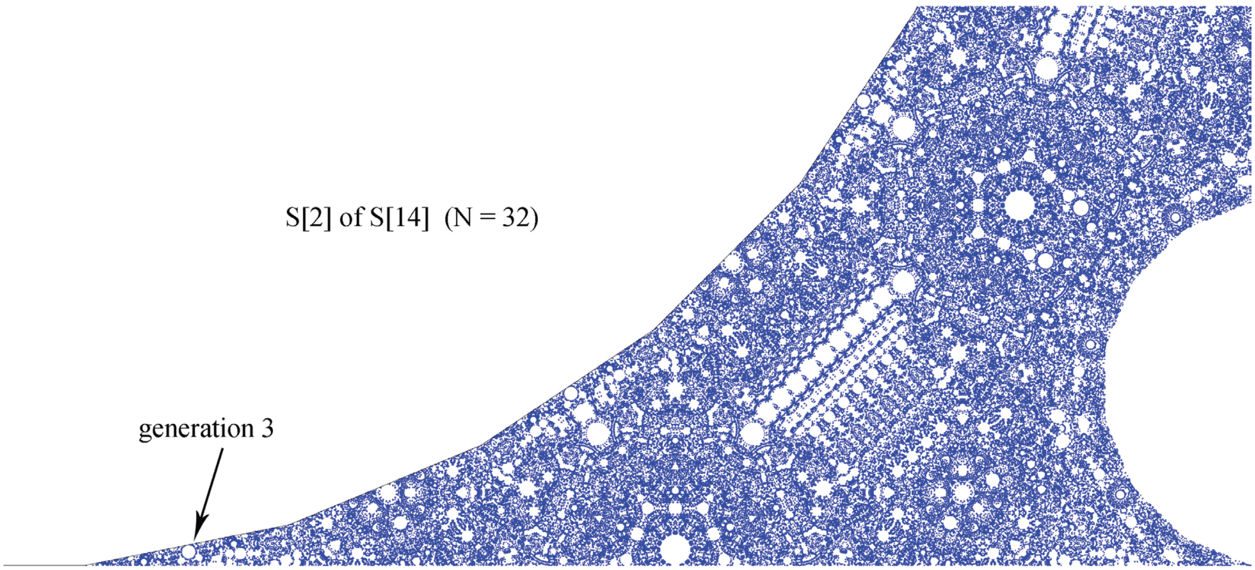
Below is an enlargement of the region at the foot of S[14]. If S[14] supports traditional generations, this would be the second generation with S[2] as patriarch. But there is no matching canonical S[1] tile.



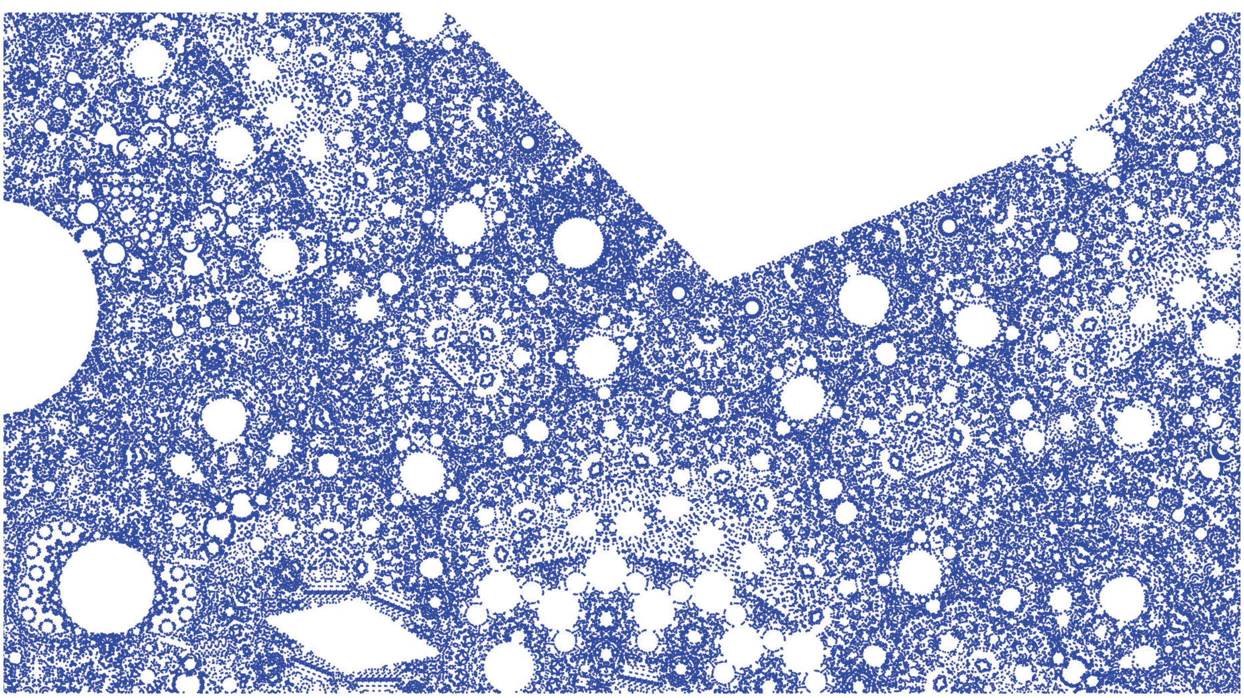
The generation 2 region is enlarged below – followed by a series of further enlargements. If there is a generation 3, it would be expected to appear at the foot of S[2] – but that region appears to be barren. The three enlargements given below cover the foot of S[2], the central region and the star[1] region. (Note the mutated octagons which are similar to S[8] and S[4][2].)



Below is the foot of S[2]. The generation 3 region shows no sign of self-similarity.

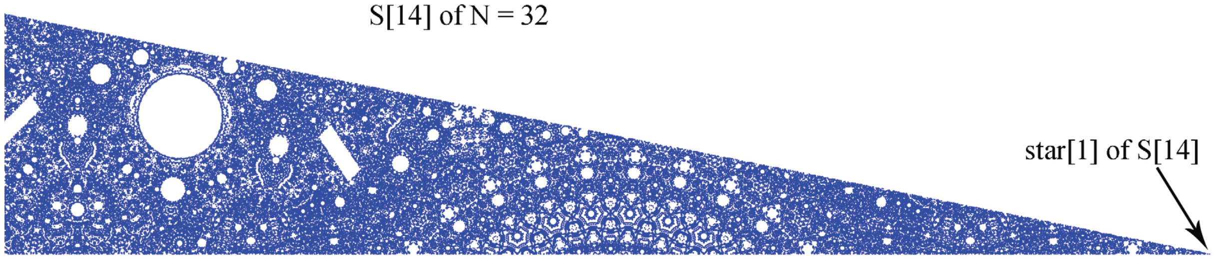


The central region of the second generation is shown below.



If S[14] supported chains of generations, there should be signs of self-similarity at the local star[1] point – but at this scale there is no obvious self-similar structure.

S[14] of N = 32



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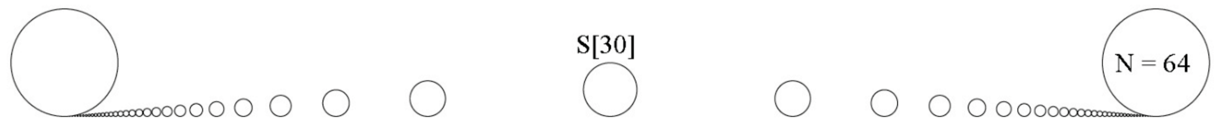
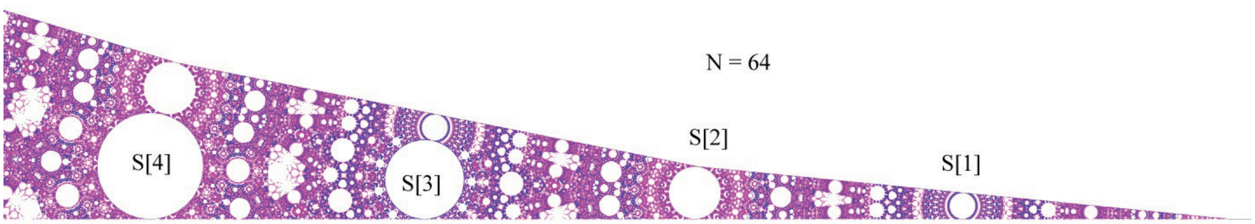
Summary of dynamics of N = 64

N = 64 is the fifth member of the 2^k family of regular polygons. Since this is a twice-even family, the best hopes for self-similar generation structure is found in the S1,S2 relationship, with S1[k] playing the role of M[k] and S2[k] playing the role of D[k]. This appears to be the case for N = 8 and N = 16. But we have found no evidence of these sequences for N = 32.

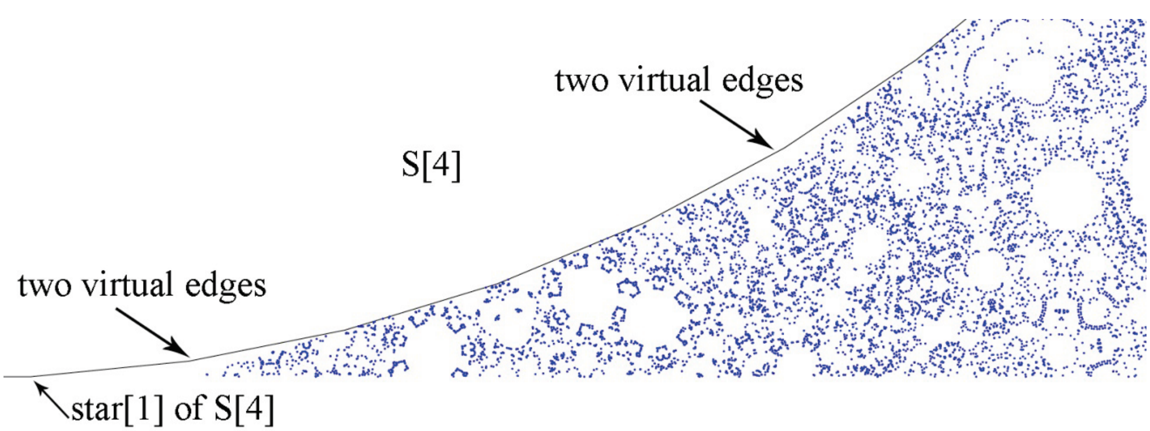
The canonical First Family is shown below. The obvious mutations occur with the S4, S8 and S[16], but there may be other more subtle mutations for S[k] tiles where GCD[k,64] >1.



Below is a portion of the First Family. Even on close inspection it is difficult to see any mutation in S[4], but it does have the expected mutation as shown in the enlargement below.



The black outline below is the canonical S[4] generated by Mathematica, but the actual S[4] has 'incomplete' edges in a step-4 format.

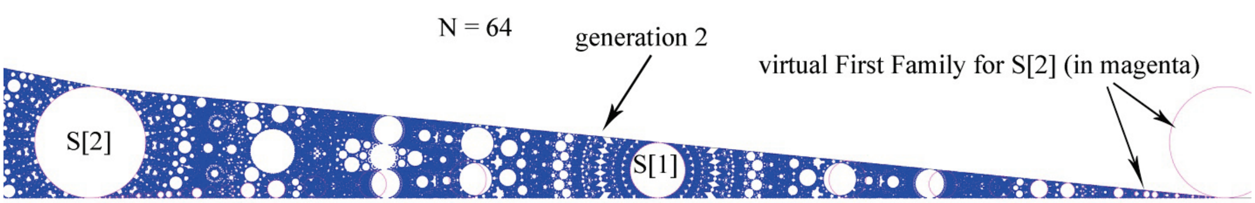


Below is the region around S[8]. At this scale the only obvious mutation is S[8] but S[6] and S[10] also have shortened periods - so they have the potential for incomplete edges.

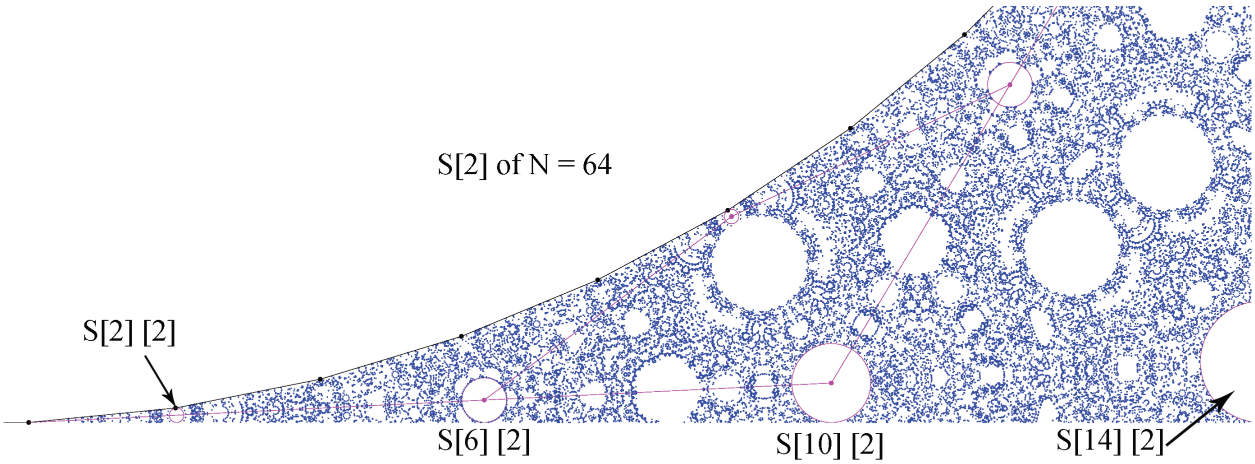
The table below shows a sample of the modified periods.

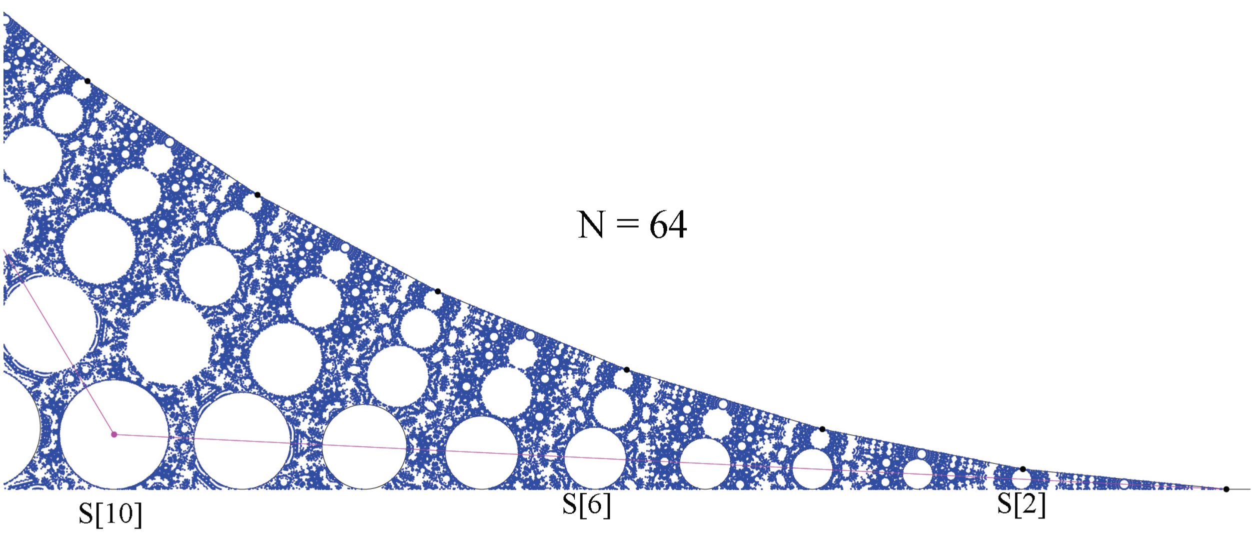
Tile	S[1]	S[2]	S[4]	S[6]	S[8]	S[10]	S[12]	S[16]	S[24]
Period	64	64	16	32	8	32	16	4	8

Below is the second generation under N = 64 – with the virtual First Family for S[2] in magenta. There are a number of matches with real tiles and this is promising. Note the symmetry between star[1] of S[2] and star[1] of N = 64. This symmetry does not survive in the third generation, but S[2] [2] survives and it supports a regular pattern of sub-tiles.



Below is the generation 3 region at star[1] of S[2]. The magenta S[2] [2] tile is real but the matching S[1] [2] does not exist so there is no traditional S[1]-S[2] family structure. However S[2] [2] does preserve some of the even-step rotational symmetry as shown below. Compare this with the next graphic which shows this generic even-step symmetry for N = 64.

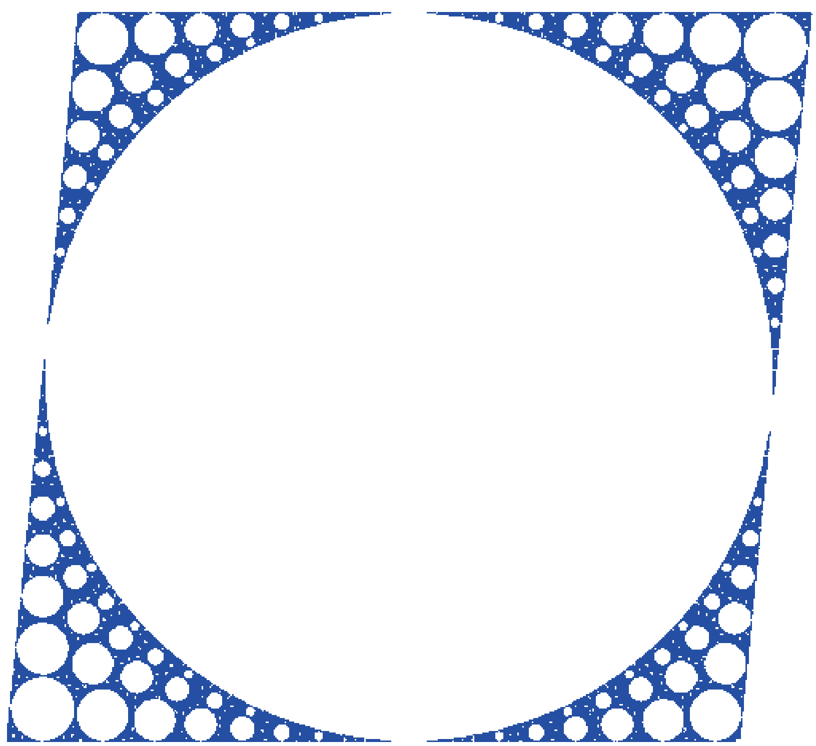




The S[30] tile

The possible Digital Filter step sizes for N = 64 range from step-1 to step-15. Most of these are totally unexplored, but the maximal step-15 web reproduces the local dynamics of the central S[30] tile. As N increases this tile becomes more isolated from its neighbors – and hence it has a large extended family.

Using the Digital Filter map with parameter: $\theta = 2\pi(15/64)$ the bounds of the torus include the first nine members of the 'First Family' for S[30] as shown below. The web shown here is identical to the Tangent Map web local to S[30] – but it can be explored in a much more efficient fashion using the Df map. Of course the members of this First Family are generally not congruent to the First Family for N = 64. Typically S[2] is the only 'survivor' as this S[30] is scaled up to have height of 1 – just like N = 64.



Below is an enlargement of the star 1 region for S[30] – showing S[2] which is identical in size and location to S[2] for N = 64 – but the local dynamics are very different. The large tile on the left is also ‘step-2’ but for S[30] there are ‘even’ and ‘odd’ step-2 orbits.