GeneticsOfPolygons.org

Siegel disks and small divisors

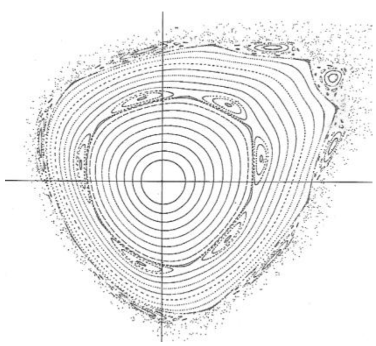
In the early part of the 20th century, Henri Poincare and George Birkoff pioneered the qualitative study of planar maps because of their applications to physical process such as celestral mechanics and quantum mechanics. Because of this framework, it was possible to prove results about Hamiltonian systems in the context of symplectic (area preserving) mappings of the plane

In terms of symplectic mappings of the plane, the issue was whether a fixed point of a 'well-behaved' mapping could remain stable when subjected to periodic disturbances. This is called the 'function theoretic center problem' because the only non-trivial case was when the fixed point was a 'center' - a fixed point which is neither attracting or repelling. Poincare knew that in these cases, the linearization process might fail because of small divisors. He had shown earlier that traditional methods of solution for the n- body problem using integrals of motion, would not work in cases like this because there is always the potential for instability.

Poincare was equally comfortable with real or complex analysis and he made fundamental contributions in both areas. He pioneered the new science of topology which provided a common framework. He formulated an 'idealized' center problem in complex analysis as follow:

Given a holomorphic (complex analytic) function such as $f(z) = \rho z + z^2$, *with* $\rho = e^{i2\pi\theta}$, *for what values of* θ *will* f *be linearizible near the fixed point* $z = 0$ *?*

A function such as f is 'linearizible' near a fixed point if it is conjugate to a rotation via a holomorphic change of coordinates. The corresponding linear map will have the form $L(z) = ze^{i2\pi\theta}$ so it will be a rotation by θ. A finite measue of such maps would guarantee local stability.



Note that in the example above, $f'(0) = \rho$, and $|\rho| = 1$, so $z = 0$ is a 'center' or indifferent fixed point. Poincare knew that when θ was rational, $|\rho - \rho^k|$ will vanish for some k (the small divisor problem) and there was little hope of convergence, so the only issue is the case when θ is irrational. In this case $z = 0$ is called an irrationally indifferent fixed point (or holomorphic germ)

The complex analysis version of the center problem has a long history and it was only recently been settled by A. Brjuno and J. Yoccoz using a class of irrationals of Brjuno type. In 1917 G.Pfeifer gave a counterexample which showed that some irrational values of θ are not linearizible. Gaston Julia (1893-1978) and Pierre Fatou (1878-1929) were working on this problem from a different perspective. For a given rational complex valued function f, they wanted to know which points are 'tame' under iteration and which points are 'exceptional' (chaotic). The latter points make up what we now call the Julia set of *f* ,and the former are the Fatou set. Neither Fatou or Julia knew how to classify the irrationally indifferent points and in 1919 Julia gave an incorrect proof that these points were never linearizable. In 1927 H. Cremer settle the case of rational θ by showing that they are not linearizable. We will see an example of this below.

The big breakthrough came in 1942 when Carl Siegel showed that if θ satisfied a certain Diophantine condition then it was linearizable. The condition guaranteed that θ could not be approximated closely by rationals.

Jurgen Moser was a friend and colleague of Siegel and in the early 1950's they both followed the developments in Russia as Kolmogorov claimed to have a proof of the stability problem. When they realized that Kolmogorov only had an outline of a proof, Moser worked on his own proof. He did not follow the path of Siegel's 1942 proof, because he knew that complex analytic functions were too 'well-behaved' to be applicable to Hamiltonian problems. His Twist Theroem assumed a real-valued 'smooth' function, while V.I. Arnold (1937-2010) assumed a real analytic function in his proof of Kolmogorov's 'conjecture'. (In the complex plane the distinction between analytic and smooth disappears.)

Any complex polynomial can be used to illustrate Siegel's result. Suppose $f(z) = z^2 + c$. Then the fixed point is no longer $z = 0$, but we can find it easily by solving $f(z) = z$. The solution depends on c. Suppose c is chosen to yield an indifferent fixed point z_0 with $|\rho| = |f'(z_0)| = 1$. We can assume that ρ has the form $e^{i2\pi\alpha}$ and normally this would imply that the motion is locally conjgate to a rotation by α, but when α is close to being rational this conjugacy might fail in any neighborhood of z_0 because the power series expansion has terms in the denominator of the form $|\rho^k - \rho|$. If $\alpha \approx m/n$, this term could get arbitrarily small and this could happen even if α was irrational.

Siegel developed a Diophantine condition on α that would guarantee that the power series converged: There exists $\varepsilon > 0$ and $\mu > 0$ such that $\left|\alpha - \frac{m}{n}\right| > \frac{\varepsilon}{n^{\mu}}$ for all integers m,n with n positive. By this measure of irrationality, the safest possible number is one with a continued fraction expansion of 0,1,1,1,.. and this number is the Golden Mean $\alpha = \gamma = \left(\sqrt{5} - 1\right)/2$

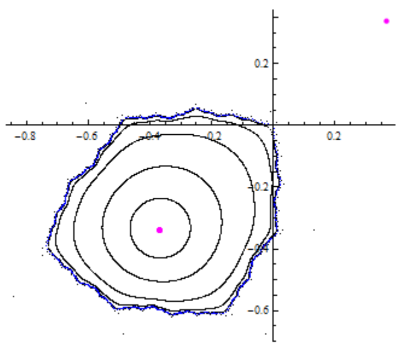
It is no coincidence that the last surviving curve of the Standard Map has winding number γ. It appears that any irrational with a continued fraction expansion which ends in 1's (a 'noble' prime) is a local 'survivor' while its neighbors are destroyed. This makes the Golden Mean the ultimate survivor.

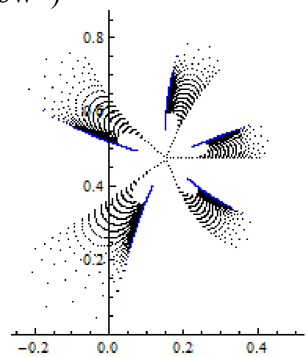
When $f(z) = z^2 + c$, $f'(z) = 2z$ so we can find a Golden Mean center z_0 by setting $2z_0 = \rho = e^{i2\pi\alpha}$ with $\alpha = \gamma$. In Mathematica: **gamma = (Sqrt[5]-1)/2;**

z0 = (1/2)*Exp[2*Pi*I*gamma] = -0.36868443903915993 - 0.3377451471307618*I (shown below)

Since this is a fixed point, z_0^2 + c0 = z_0 so c0 = -0.3905408702184 - 0.5867879073469687*I
Mathematica does not care whether a function is real or complex, so define: **f[z_]:= z^2 + c0** and iterate *f* in the neighborhood of z_0:

Scan =Flatten[Table[NestList[f, z0+ x, 1000],{x, 0, .55, .1}] ,1];
Two special orbits: **Orbit1 =NestList[f, 0, 3000]; Orbit2 =NestList[f, -z0, 1];** (*in blue*)
To plot a complex valued point: **CPoint[z_]:=Point[{Re[z],Im[z]}];**
Graphics[{ CPoint/@Scan, Blue, CPoint/@Orbit1, AbsolutePointSize[5.0], Magenta, CPoint/@Orbit2}, Axes->True] (*on the left below*)





The invariant curves shown on the left are the complex analog of invariant KAM curves. (There are no invariant curves on the right hand plot because it corresponds to gamma = 1/5. This is a rationally indifferent fixed point and there are no stable regions of any size so linearization is impossible. The divergent channels seen here extend all the way to z0. Julia knew this and it is one reason why he was so pessimistic about the irrational case.)

Returning to the main plot, the 2 magenta points show that the symmetric point -z0 maps to z0 so by continuity the neighborhood of -z0 maps to the plot above. In both plots the blue orbit is the orbit of z = 0. This is always a 'critical' point of *f*, since the derivative vanishes there. On the left we can see that the blue orbit forms the boundary of the invariant region. The interior of this region is called the Siegel Disk of z0. The boundary curve is a continuum but it is never (real) analytic. However this does not prevent it from possibly being smooth. (No examples of this are known for quadratic polynomials.) As expected, the Siegel Disk is in the Fatou set of *f* and the boundary points are in the Julia set.

Fatou and Julia knew that for rational complex functions, the orbit of the critical point can be used to characterize the dynamics.

Definition: The Julia set, Jc of $f(z)= z^2+ c$ is the closure of the set of repelling periodic points. (A point p of period k is repelling if $|f^k\,'(p)| > 1$.)

Thus J_c consists of all repelling periodic points and their limit points. An equivalent characterization is: Jc is the boundary of the set of points which diverge under f^k.

Theorem (P. Fatou & G. Julia, 1918) Let Ω denote the set of critical points for a polynomial function *f* and let K denote the set of points which do not diverge to ∞

(i) $\Omega \subseteq K \Leftrightarrow$ J is connnected

(ii) $\Omega \cap K = \varnothing \Leftrightarrow$ J is a Cantor set

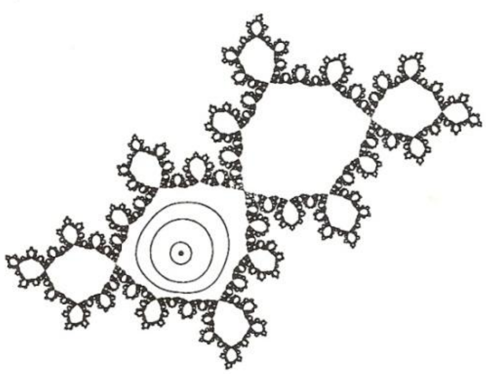
A polynomial such as $f(z) = z^2 + c$ will always have z = 0 as a critical point and z = ∞ is another critical point - but it is not very interesting because it is always attracting. This leaves z = 0 as the only interesting point. If it does not diverge than J is connected

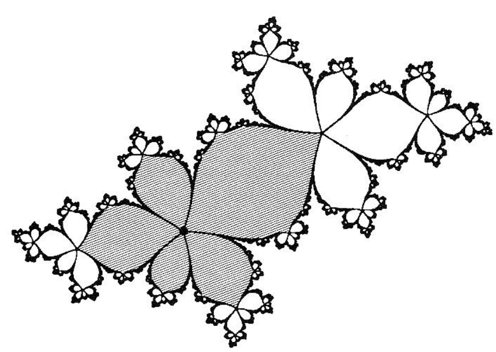
Definition:The Mandelbrot set $M = \{c \in \mathbb{C} : Jc \text{ is connected}\} = \{c \in \mathbb{C} : f^k(0) \text{ does not diverge}\}$

It appears from the blue orbit above, that z = 0 does not diverge so c_0 is in M, and it must be on the boundary because the interior of M consists of c values where the fixed point is attracting. If c corresponded to a rationally indifferent fixed point, such as the period 5 case above, it would be the cusp of a period 5 bud on the boundary of M. (These are also called parabolic points). The only other possibility for the Golden Mean case is a boundary point which is not a cusp.

Below are the two Julia sets (complements of Peitgen and Richter). Both are (marginally) connected, but in the rational case the chaotic dynamics reach all the way into the fixed point z0. For the Golden Mean case on left, the invariant curves around z0 are a sign of stability - just as they are for the KAM Theorem.These invariant curves are shown here just for reference- they are not part of the Julia set - which is the chaotic boundary.

The Siegel disk boundary forms part of the Jilia set and the rest are preimages. (In this sense the Julia set is the 'web' defined by the Siegel Disk as the generating 'polygon'.) A typical point in one of the smaller buds would progress from smaller buds to larger buds and finally it would be mapped to one of the invariant curves. For polynomial functions, M. Herman proved that the Julia set always contains the critical point so z = 0 is in the Julia set and if we zoom into this region it has a self-similar fractal boundary.





Theorem (C. Siegel, A. Brjuno and J. Yoccoz). A quadratic polynomial *f* is linearizable near 0 if and only if $\sum_{n-1}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty$ where p_n/q_n are the rational convergents of θ.

The work of Siegel, Brjuno and Yoccoz was concerned entirely with the local dynamics at z0 and very little is known about the global dynamics in the region of a Siegel disk D. The boundary forms a non-analytic continuum but is it a Jordan curve ? It is possible that for some Siegel disks, the boundary might be an indecomposable continuum. On the other extreme there are rotation numbers θ which yield smooth boundaries - but no examples are known for quadratic polynomials.