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# Summary of dynamics of the regular octagon: $N = 8$

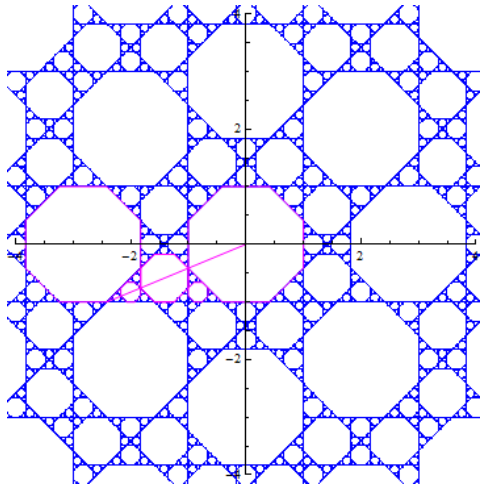


```

web[.01,5,300,0];WebPlot={};For[i=1,i<=npoints,i++,WebPlot = Union[WebPlot,Jxy[[i]]];
box[{0,0},4]; (*crop region for the plot below*)

Show[Graphics[{AbsolutePointSize[1.0],Blue,Point[WebPlot],poly[Mom],Magenta,poly/@
FirstFamily, Line[{0,0},GenStar]}],Axes->True,PlotRange->{{left,right},{bottom,top}}]]

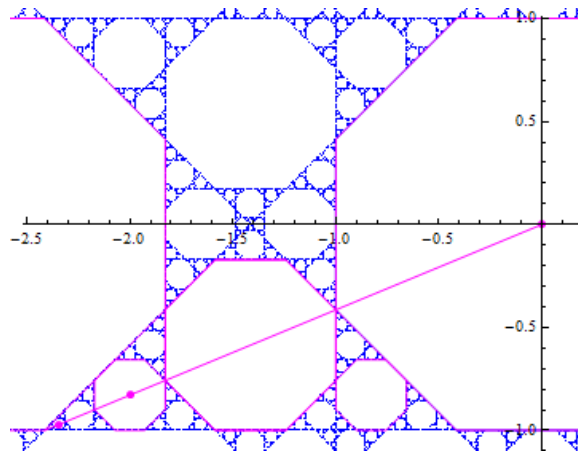
```



The central region inside the ring of 8 Dads is invariant and serves as a template for the global dynamics. The line of symmetry shown above runs from the origin to GenStar. There is a chain of generations of Dads converging to GenStar and their centers all lie on this line. We can find these centers because `GenerationScale[8]` gives the scale for each generation relative to Dad, so the height of  $Dad[k] = (\text{height of Dad}) * \text{GenScale}^k = \text{GenScale}^k$

Center from height =  $CFR[r_] := (1-h)*GenStar$ ; center of  $Dad[k] = CFR[GenScale^k]$ .

The first two centers in this chain are shown below (actually  $\{0,0\}$  is the first member).

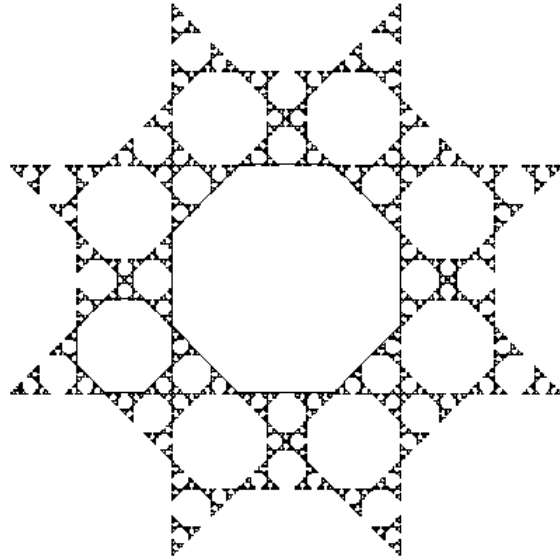


The ratio of the periods of the Dads approaches 9: The first 7 terms in the sequence are of periods are 16, 96, 1008, 8640, 79056, 707616, 6382208,.. By the time we reach  $Dad[7]$  the

orbits are virtually non-periodic and their density is very uniform. Below is the first 50,000 points in the orbit of  $Dad[7]$ :

$$cDad[7] = CFR[GenScale^7] \approx \{-2.4142029962396670, -0.9999956233642322980185191\}$$

$Orbit = V[cDad[7], 50000]; Graphics[{\text{AbsolutePointSize}[1.0], \text{Point}[Orbit]}$  (\*about 20 seconds\*)



This implies that the fractal dimension of the 'web' is  $\text{Ln}[9]/\text{Ln}[1/GenScale] \approx 1.24648$  which is close to  $N = 5$  at  $\text{Ln}[6]/\text{Ln}[1/GenScale[5]] \approx 1.24114$ . They are both sparse compared to the classic Sierpinski triangle at  $\text{Ln}[3]/\text{Ln}[2] \approx 1.58496$ .

All the regular  $N$ -gons of the form  $N = 2^k$  are constructible because they can be constructed from the square by successive angle bisections. If  $N$  is one of these 'powers of 2' regular polygons, then  $\varphi(N)$  (EulerPhi) =  $N/2$  and the degree of the minimal polynomial for  $\text{Cos}[2\text{Pi}/N]$  is always half of this.

For  $N = 8$ ,  $\varphi(8) = 4$  and  $\text{MinimalPolynomial}[\text{Cos}[2*\text{Pi}/8]][x] = 2x^2 - 1$  so  $\text{Cos}[\text{Pi}/4] = 1/\sqrt{2}$

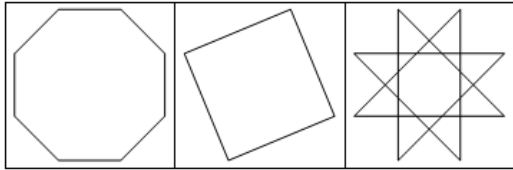
This implies that the vertex set lies in  $\mathbb{Z}(\sqrt{2})$  in a fashion similar to the regular pentagon. It is no coincidence that the webs for both of these are self-similar fractals.

## Projections

$N = 8$  has only one non-trivial projection but our convention will be to show all the possible projections. In this case the 'redundant' P2 projection is also interesting. The projections clearly show the self-similar nature of the orbits. Each orbit is a 'refinement' of the previous.

The three possible mappings of the vertices are stored in the matrices  $Wc[[1]]$ ,  $Wc[[2]]$  and  $Wc[[3]]$ . These are shown below

**GraphicsGrid[Table[Graphics[poly[Wc[[k]]]],{k,1,HalfN-1}],Frame->All]**



Below we will show the projections for the sequence of Dads

Example 1:  $cDad[1] = CFR[GenScale] \approx \{-2.0, -0.828427124746190097603\}$

This point is period 16, so the projections will have period 8

**Ind = IND[q1,20];** (\*generate first 20 vertices in the orbit\*)

**k = 9;** (\*Plot one more than half the period to close the plot\*)

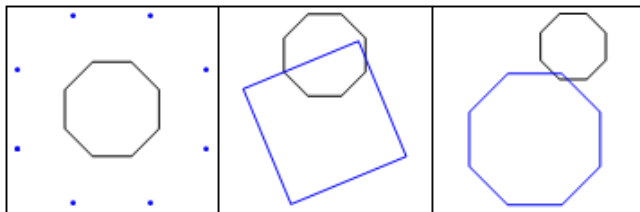
**Px=Table[Graphics[{poly[Mom],Blue,Line[PIM[q1,k,j]]}],{j,1,HalfN-1}];**

(\*generate the three projection using pairs of vertices from Ind\*)

**Px[[1]]=Graphics[{poly[Mom],AbsolutePointSize[3.0],Blue,Point[PIM[q1,k,1]]}];**

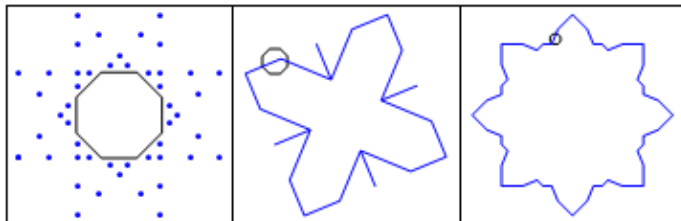
(\*plot the P1 projection with points instead of lines\*)

**GraphicsGrid[{{Px[[1]],Px[[2]],Px[[3]]},Frame->All]**

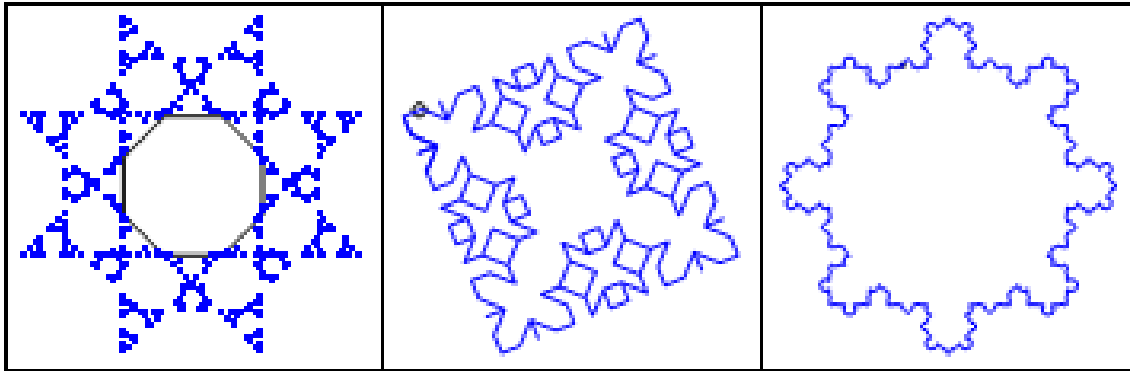


Example 2:  $cDad[2] = CFR[GenScale^2] \approx \{-2.34314575050761, -0.970562748477140585\}$

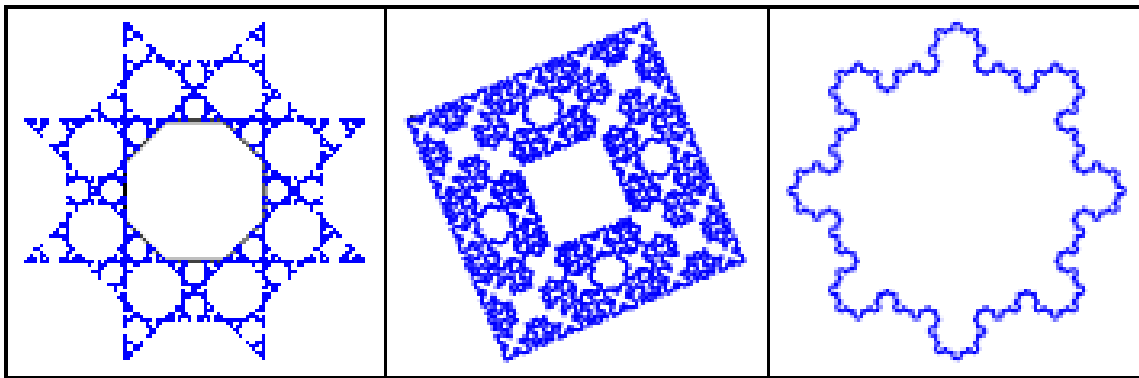
Period 96 so **k = 49;**



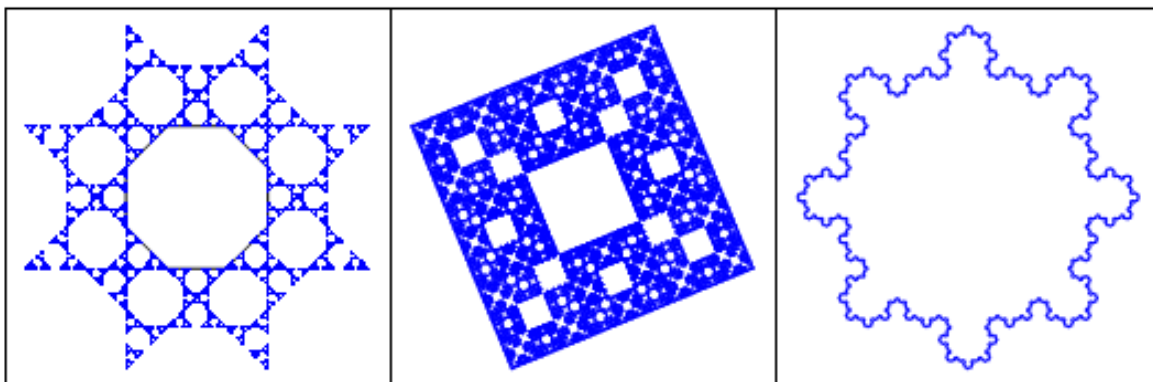
Example 3:  $cDad[3] = CFR[GenScale^3] \approx \{-2.4020202535533, -0.994949366116653416\}$   
Period 1008 so  $k = 505$ ;



Example 4:  $cDad[4] = CFR[GenScale^4] \approx \{-2.41212152132003, -0.99913344822277991108\}$   
Period 8640 so  $k = 4321$ ;



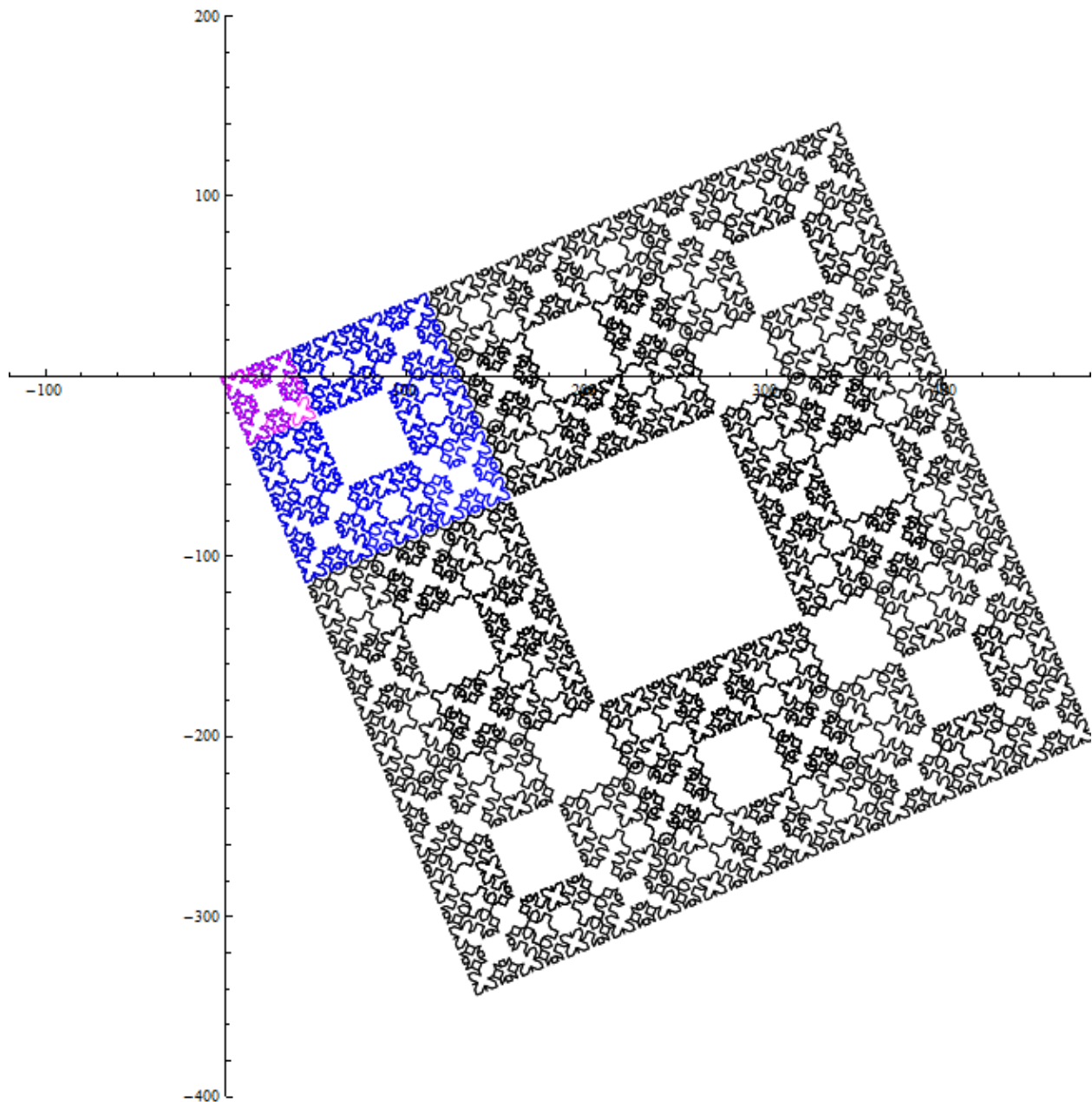
Example 5:  $cDad[5] = CFR[GenScale^5] \approx \{-2.413854624874471, -0.9998513232200260505\}$   
Period 79056 so  $k = 39529$ ;



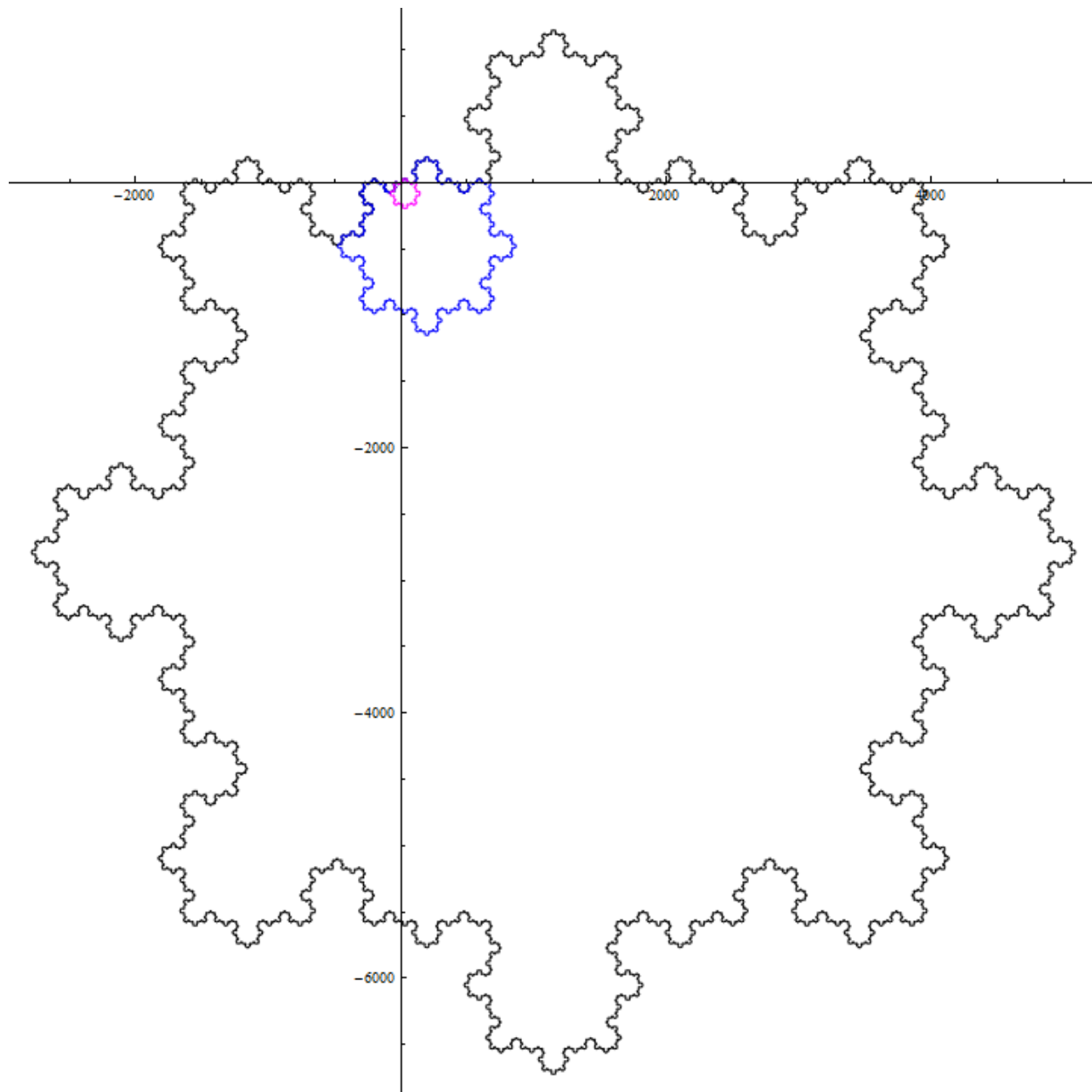
These projections are increasing in scale and they are 'nested' . To see this we will plot the last three in different colors below using Dad[3] (magenta) ,Dad[4] (blue) and Dad[5] (black):

```
Ind= IND[cDad[3],2000]; Px32= PIM[cDad[3],520,2]; Px33=PIM[cDad[3],520,3];  
Ind= IND[cDad[4],10000]; Px42= PIM[cDad[4],4500,2]; Px43=PIM[cDad[3],4500,3];  
Ind= IND[cDad[5],100000]; Px52= PIM[cDad[5],40000,2]; Px53=PIM[cDad[3],40000,3];
```

```
Graphics[{AbsolutePointSize[1.0],Line[Px52], Blue, Line[Px42],Magenta,Line[Px32]},Axes->True]
```

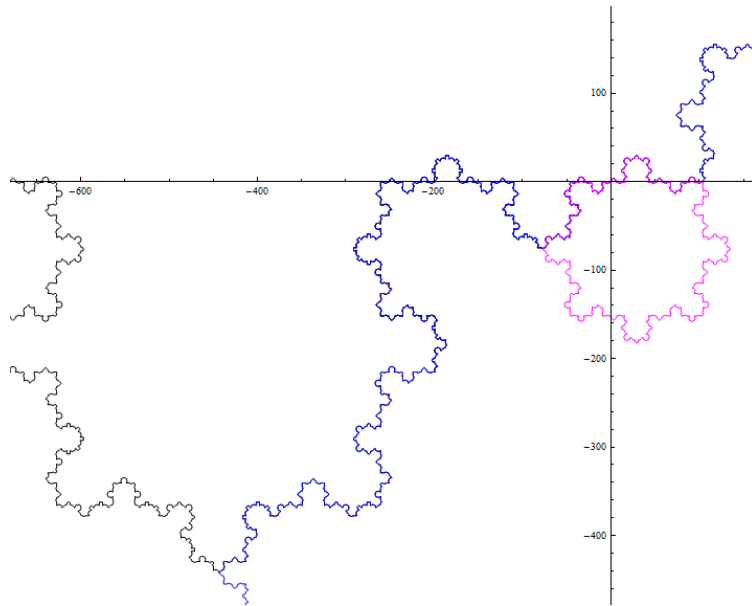


`Graphics[{AbsolutePointSize[1.0],Line[Px53], Blue, Line[Px43],Magenta,Line[Px33]},Axes->True]`

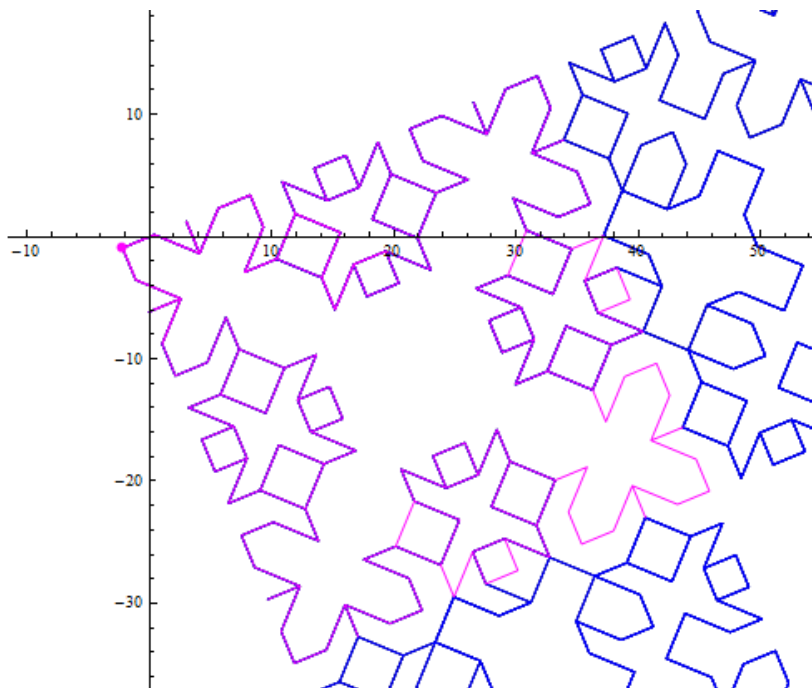




This chain is clearly endless. Since the periods grow by a factor which approaches 9 we can renormalize this chain to turn this large-scale fractal structure into a Koch snowflake whose fractal dimension should be  $\text{Ln}[9]/\text{Ln}[1/\text{GenScale}] \approx 1.24648$ . The motion is clockwise. Below is an enlargement of the region around Mom. Note that the Blue and Black branch points are self-similar



Below is a similar enlargement of the 'sponge'. In the limit it should have the same fractal dimension as the snowflake above. cDad[3] is shown as a Magenta dot. The dynamics here are complicated by the redundancy so the branching is highly non-trivial.



Richard Schwartz at Brown University obtained similar plots using Pinwheel projections. The projections shown above are not Pinwheel projections but they are a type of Arithmetic Graph that has proven useful in his quest to track unbounded orbits. For more information see the section on Projections, Pinwheel Maps and Arithmetic Graphs.

### **References**

Schwartz R.E. , Unbounded Orbits for Outer Billiards, *Journal of Modern Dynamics* 3 (2007)

Schwartz R.E. , Outer Billiards on Kites, *Annals of Mathematics Studies*, 171 (2009), Princeton University Press, ISBN 978-0-691-14249-4