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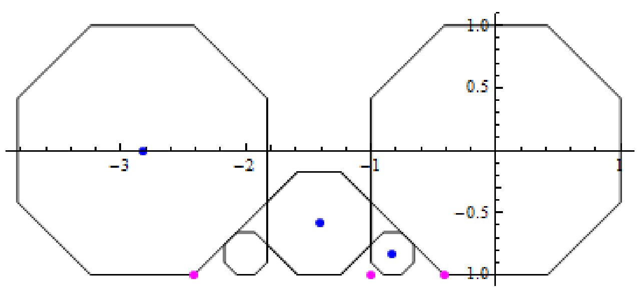
Summary of dynamics of the regular octagon: N = 8

N = 8 is the second 'twice-even' regular polygon. It is also part of the 'powers of 2' family of polygons. Unlike the 'twice-odd' families, there is typically no relationship between N = 2k and N = k, so the dynamics of N = 8 are largely unrelated to the dynamics of N = 4, and N = 16 bears little or no relationship with N = 8.

For regular N-gons with N even, our convention is to use a height of 1 instead of a radius of 1. We will also rotate them to orientate them 'upright'. This put vertex 1 at top right. In the notebook NTwiceEven.nb, the First Family for N = 8 is derived from the 'star' points as explained in N16Summary. That First Family is shown below along with the star points and centers..

Graphics[{poly/@FirstFamily, AbsolutePointSize[5.0], Magenta, Point[{star[[1]],star[[2]],star[[3]]}], Blue, Point[{cS[1],cS[2],cS[3]}]}, Axes->True]



The horizontal displacements between the star points and the centers is always s0/2 where s0 is the side of Dad. This means that star points define centers and conversely. This is true for all regular polygons, because the star points define the bounds for the Tangent Map. In addition each star point defines a scale which is independent of the size of Dad.

scale[k_] := s0/(-2*star[k][[1]]); (*see the page on Scaling*)

Column[Table[scale[k], {k,1,HalfN-1}]]
1.000000000000000000000000000
0.414213562373095048801688724 21
0.171572875253809902396622551 58

scale[3] ≈ 0.17157287525380990239662255158 is also known as **GenerationScale[8]** which we sometimes shorten to GenScale. The star point that corresponds to GenScale is known as GenStar. This is star[[3]] on the left above.

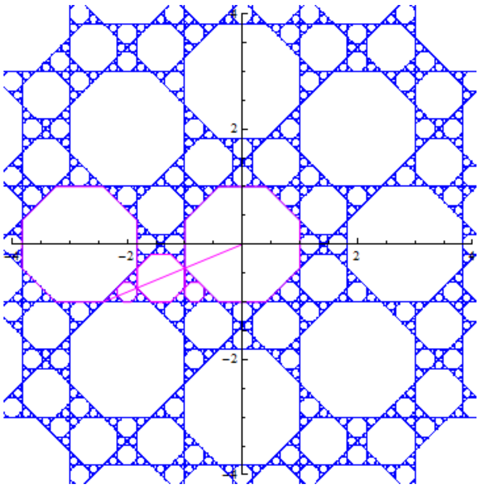
N = 8 has an infinite string of generations converging to GenStar and one of these is the remaining First Family member above. He is called LS[1] which is short for LeftS[1]. Note that LS[1] has the same size as S[1]. The large Dad on the left above is known as S[3] or simply DadLeft. His star points can be used to find the center of LS[1], or we can just rely on symmetry.

A web scan will uncover the basic dynamics of any regular polygon.

web[.01,5,300,0];WebPlot={};For[i=1,i<=npoints,i++,WebPlot = Union[WebPlot,Jxy[[i]]]];

box[{0,0},4]; (*crop region for the plot below*)

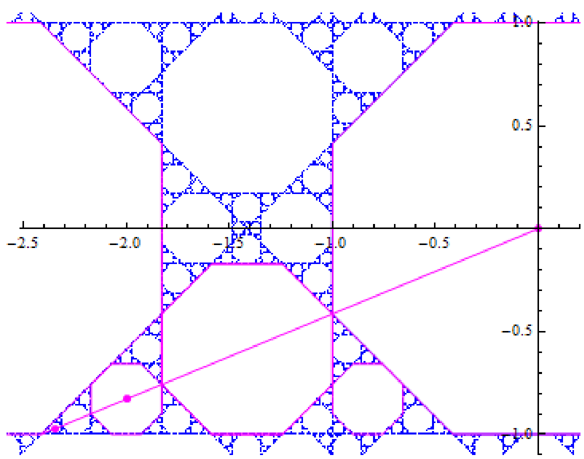
Show[Graphics[{AbsolutePointSize[1.0],Blue,Point[WebPlot],poly[M],Magenta,poly/@ FirstFamily, Line[{0,0},GenStar}]},Axes->True,PlotRange->{{left,right},{bottom,top}}]]



The central region inside the ring of 8 Ds is invariant and serves as a template for the global dynamics. The line of symmetry shown above runs from the origin to GenStar. There is a chain of generations of Ds converging to GenStar and their centers all lie on this line. We can find these centers because GenerationScale[8] gives the scale for each generation relative to D, so the height of D[k] = (height of D)*GenScale^k = GenScale^k

Center from height = **CFR[r_]:= (1-h)*GenStar;** center of D[k] = CFR[Genscale^k].

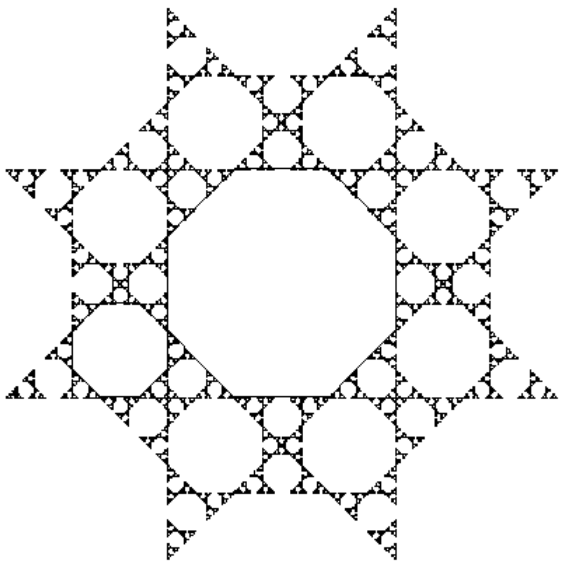
The first two centers in this chain are shown below (actually {0,0} is the first member).



The ratio of the periods of the Dads approaches 9: The first 7 terms in the sequence are of periods are 16, 96, 1008, 8640, 79056, 707616, 6382208,.. By the time we reach Dad[7] the orbits are virtually non-periodic and their density is very uniform. Below is the first 50,000 points in the orbit of D[7]:

cDad[7] = CFR[GenScale^7] ≈ {-2.4142029962396670,-0.9999956233642322980185191}

Orbit = V[cDad[7],50000];Graphics[{AbsolutePointSize[1.0],Point[Orbit]}] (*about 20 seconds*)



This implies that the fractal dimension of the 'web' is Ln[9]/Ln[1/GenScale] ≈ 1.24648 which is close to N = 5 at Ln[6]/Ln[1/GenScale[5]] ≈ 1.24114. They are both sparse compared to the classic Sierpinski triangle at Ln[3]/Ln[2] ≈1.58496.

All the regular N-gons of the form N = 2^k are constructible because they can be constructed from the square by successive angle bisections. If N is one of these ‘powers of 2’ regular polygons, then φ(N) (EulerPhi) = N/2 and the degree of the minimal polynomial for Cos[2Pi/N] is always half of this.

For N = 8, φ(8) = 4 and **MinimalPolynomial[Cos[2*Pi/8]][x]** = $2x^2$-1 so Cos[Pi/4] = $1/\sqrt{2}$

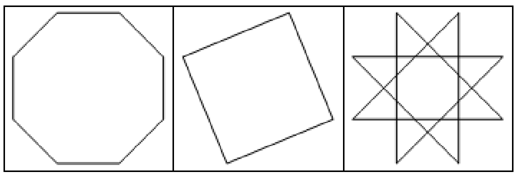
This implies that the vertex set lies in $\mathbb{Z}(\sqrt{2})$ in a fashion similar to the regular pentagon.It is no coincidence that the webs for both of these are self-similar fractals.

Projections

N = 8 has only one non-trivial projection but our convention will be to show all the possible projections. In this case the 'redundant' P2 projection is also interesting. The projections clearly show the self-similar nature of the orbits.Each orbit is a 'refinement' of the previous.

The three possible mappings of the vertices are stored in the matrices Wc[[1]], Wc[[2]] and Wc[[3]]. These are shown below

GraphicsGrid[{Table[Graphics[poly[Wc[[k]]]],{k,1,HalfN-1}]},Frame->All]



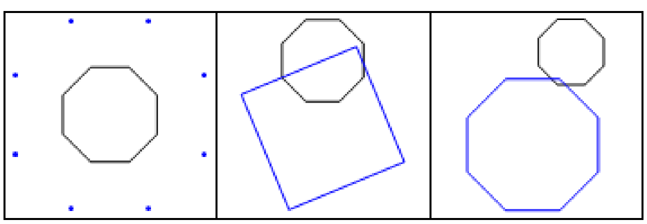
Below we will show the projections for the sequence of Ds

Example 1: cDad[1] = CFR[GenScale] ≈{-2.0,-0.828427124746190097603}
This point is period 16, so the projections will have period 8

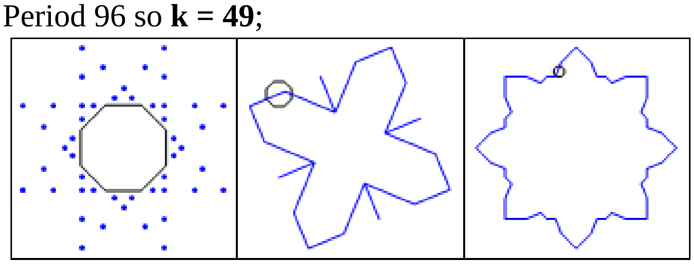
Ind = IND[q1,20]; (*generate first 20 vertices in the orbit*)
k = 9; (*Plot one more than half the period to close the plot*)

Px=Table[Graphics[{poly[Mom],Blue,Line[PIM[q1,k,j]]}],{j,1,HalfN-1}];
(*generate the three projection using pairs of vertices from Ind*)
Px[[1]]=Graphics[{poly[Mom],AbsolutePointSize[3.0],Blue,Point[PIM[q1,k,1]]}];
(*plot the P1 projection with points instead of lines*)

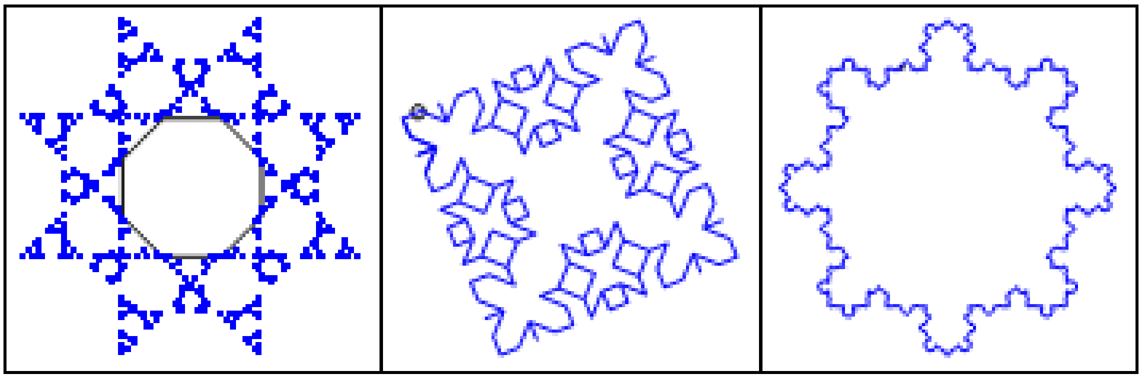
GraphicsGrid[{{Px[[1]],Px[[2]],Px[[3]]}},Frame->All]



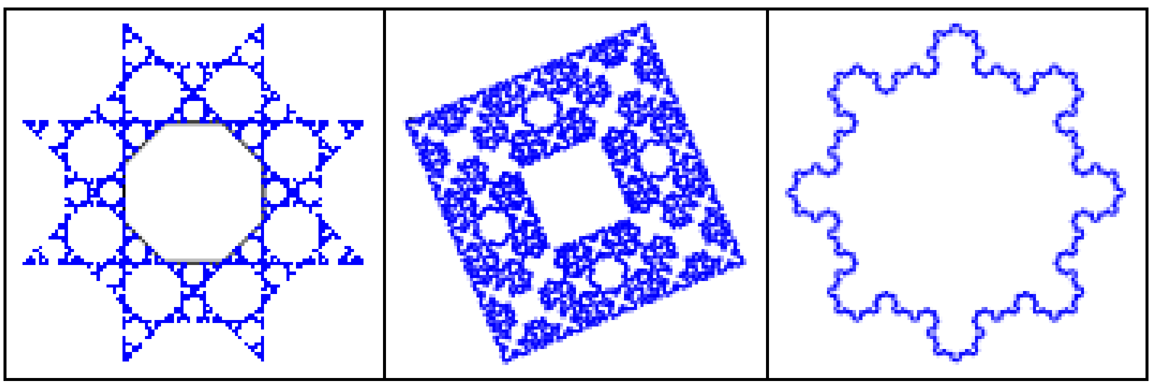
Example 2: cDad[2] = CFR[GenScale^2] ≈ {-2.34314575050761,-0.970562748477140585}
Period 96 so **k = 49**;



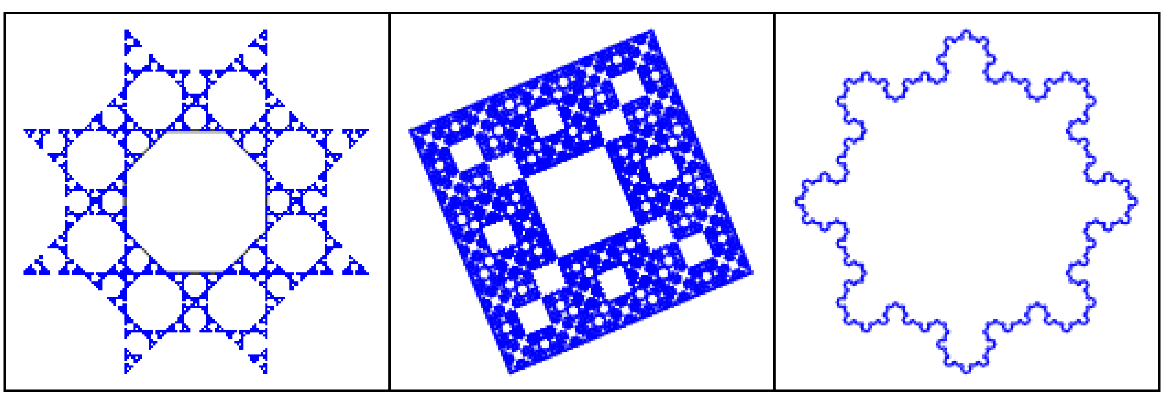
Example 3: cD[3] = CFR[GenScale^3] ≈ {-2.4020202535533,-0.994949366116653416}
Period 1008 so **k = 505;**



Example 4: cD[4] = CFR[GenScale^4] ≈{-2.41212152132003,-0.99913344822277991108}
Period 8640 so **k = 4321;**



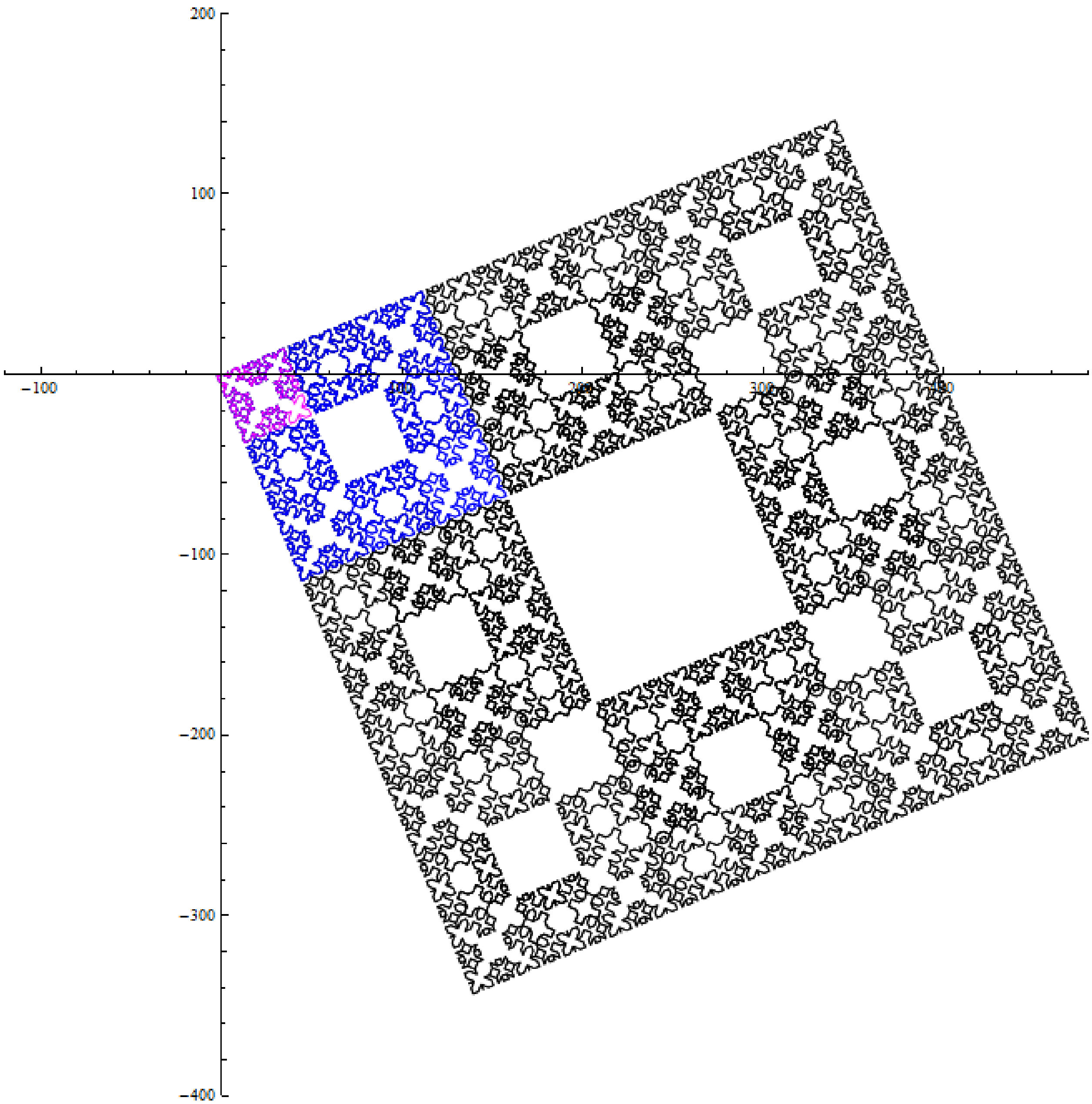
Example 5: cD[5] = CFR[GenScale^5] ≈ {-2.413854624874471,-0.99985132322200260505}
Period 79056 so **k = 39529;**



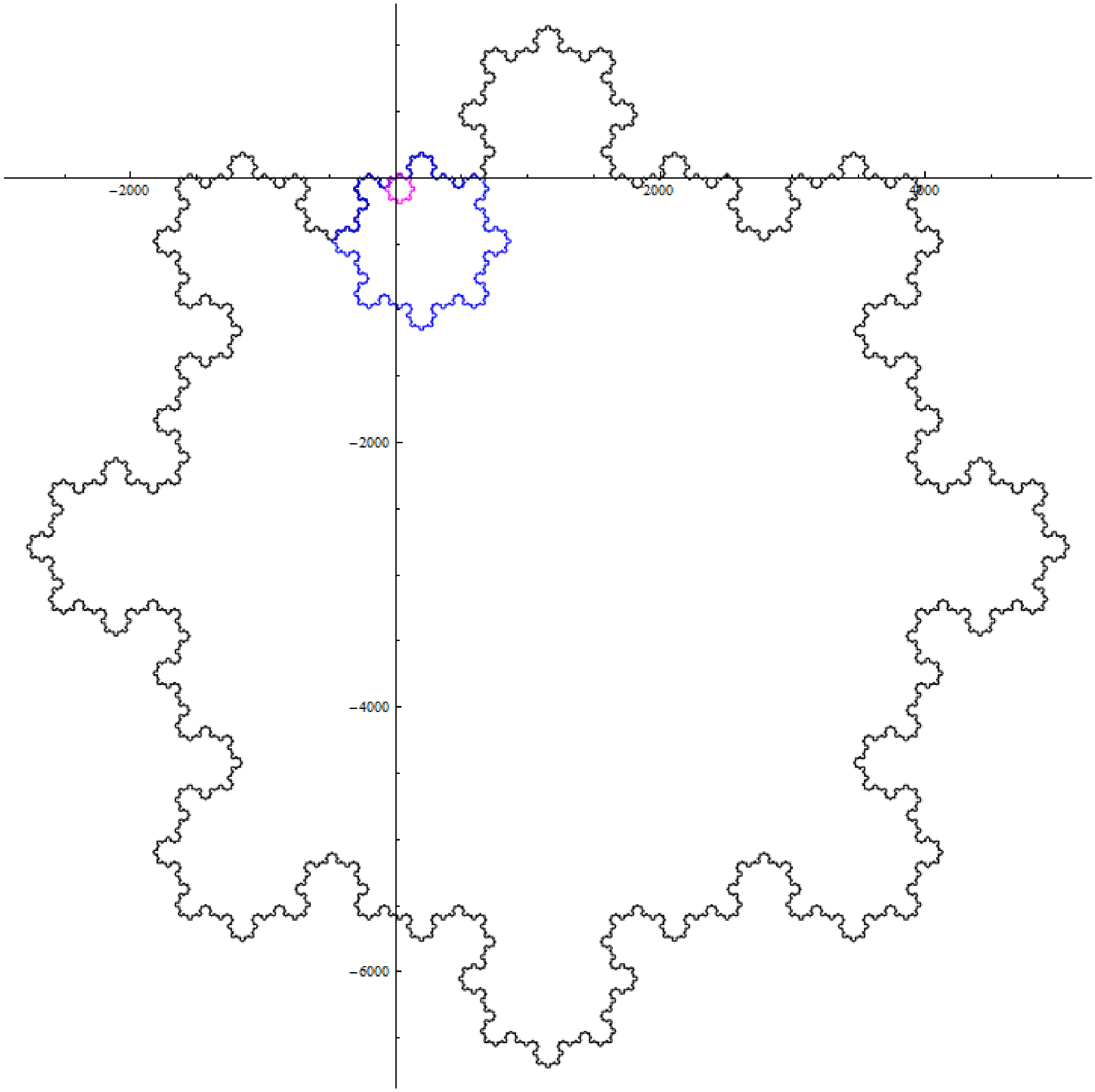
These projections are increasing in scale and they are 'nested' . To see this we will plot the last three in different colors below using D[3] (magenta) ,D[4] (blue) and Da5] (black):

Ind= IND[cDad[3],2000]; Px32= PIM[cD[3],520,2]; Px33=PIM[cD[3],520,3];
Ind= IND[cDad[4],10000]; Px42= PIM[cD[4],4500,2]; Px43=PIM[cD[3],4500,3];
Ind= IND[cDad[5],100000]; Px52= PIM[cD[5],40000,2]; Px53=PIM[cD[3],40000,3];

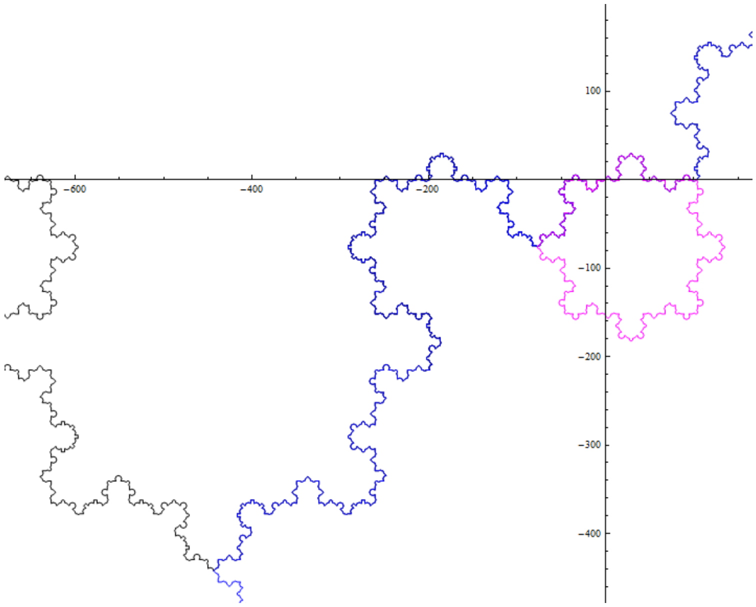
Graphics[{AbsolutePointSize[1.0],Line[Px52], Blue, Line[Px42],Magenta,Line[Px32]},Axes->True]



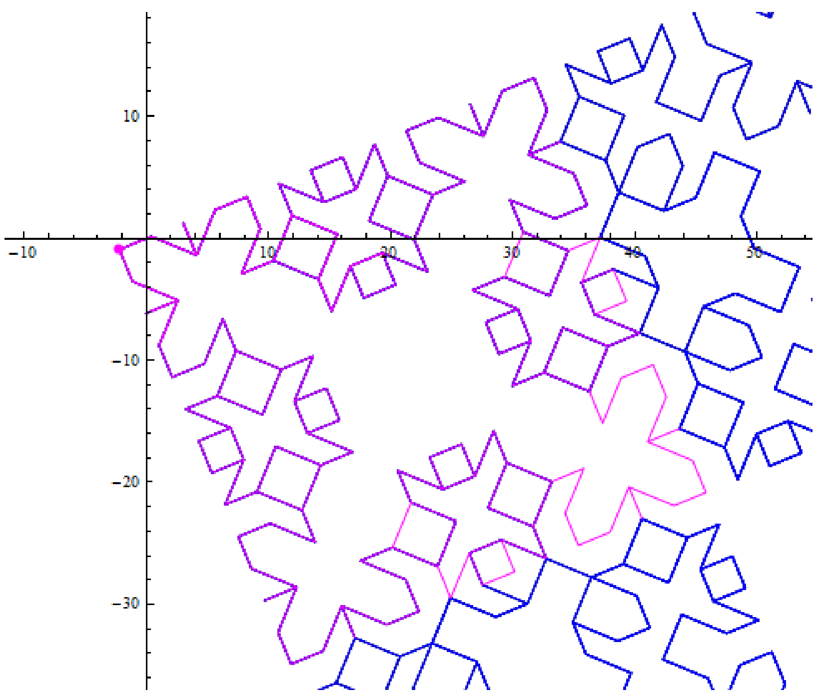
Graphics[{AbsolutePointSize[1.0],Line[Px53], Blue, Line[Px43],Magenta,Line[Px33]},Axes->True]



This chain is clearly endless. Since the periods grow by a factor which approaches 9 we can renormalize this chain to turn this large-scale fractal structure into a Koch snowflake whose fractal dimension should be Ln[9]/Ln[1/GenScale] ≈ 1.24648 . The motion is clockwise. Below is an enlargement of the region around Mom. Note that the Blue and Black branch points are self-similar



Below is a similar enlargement of the 'sponge'. In the limit it should have the same fractal dimension as the snowflake above. cD[3] is shown as a Magenta dot. The dynamics here are complicated by the redundancy so the branching is highly non-trivial.



Richard Schwartz at Brown University obtained similar plots using Pinwheel projections. The projections shown above are not Pinwheel projections but they are a type of Arithmetic Graph that has proven useful in his quest to track unbounded orbits. For more information see the section on Projections, Pinwheel Maps and Arithmetic Graphs.

References

Schwartz R.E. , Unbounded Orbits for Outer Billiards, Journal of Modern Dynamics 3 (2007)

Schwartz R.E. , Outer Billiards on Kites, Annals of Mathematics Studies, 171 (2009), Princeton University Press, ISBN 978-0-691-14249-4