Summary of dynamics of the regular heptagon: $N = 7$
Introduction

The following arXiv articles have background information about the dynamics of regular N-gons. [H5] has recently been updated and it is recommended. We will summarize some of the content here.


[H3] Hughes G.H., Outer billiards on Regular Polygons, arXiv:1311.6763


Some of this material about N = 7 spans a period of 20 years. The Wikipedia article on ‘heptagons’ has a reference to my article on Polygons of Albrecht Dürer at arXiv:1205.0080. The heptagon is the first regular case that cannot be constructed by compass and straight edge so Durer’s construction is only approximate – and he probably knew this. See constructions.pdf for more on this issue.

Here is what will come up if you run FirstFamily.nb with npoints set to N = 7

The First Family of N = 7

The three S[k] tiles form the ‘nucleus’of the First Family. S[3] is also known as D. It is a regular 14-gon with same side as N = 7. The First Family of N = 7 consists of this nucleus together with the First Family of D, so the rest of the First Family are called DS[1], DS[2] and DS[3]. DS[4] is the same as DS[2] of N = 7.
Below is how this First Family is embedded in the star polygons of N = 7

MatrixForm[Table[-star[[k]][[[1]], {k, 1, 3}]]] will give the horizontal coordinates of the star points. By default Mathematica converts Tan to Cot after Pi/7 so Tan[2*Pi/7] is Cot[3*Pi/14]

\[
\begin{pmatrix}
\tan\left(\frac{\pi}{7}\right) \\
\cot\left(\frac{3\pi}{14}\right) \\
\cot\left(\frac{\pi}{14}\right)
\end{pmatrix}
\]

MatrixForm[Table[scale[k], {k, 1, 3}]] will give the scales

\[
\begin{pmatrix}
\tan\left(\frac{\pi}{7}\right) & \tan\left(\frac{3\pi}{14}\right) \\
\tan\left(\frac{\pi}{14}\right) & \tan\left(\frac{\pi}{7}\right)
\end{pmatrix}
\]

MatrixForm[Table[hS[1]/hS[k], {k, 1, 3}]] is identical so the scales are the relative heights.

MatrixForm[Table[AlgebraicNumberPolynomial[ToNumberField[scale[k], GenScale], x], {k, 1, 3}]]

This gives polynomials for the three scales in terms of x = GenScale[7] = scale[3]

\[
\begin{pmatrix}
1 \\
\frac{1}{2} - x - \frac{x^2}{2} \\
x
\end{pmatrix}
\]

MatrixForm[Table[AlgebraicUnitQ[scale[k]], {k, 1, 3}]] will only report True for ‘primitive scale[k] with gcd(k,N) = 1, so here all are primitive.

NumberFieldSignature[GenScale]

(* This gives the number of real or complex roots of the minimal polynomial of GenScale. Since GenScale generates the scaling field S, this field is real and has dimension EulerPhi[N/2] so there are always EulerPhi[N/2] real roots (and no complex roots). S is a subfield of the full cyclotomic*)
field \( Q(z) \) which is generated by the complex number \( z = \text{Exp}[2*I*\text{Pi}/N] \). This field has dimension \( \text{EulerPhi}[N] \) which is twice that of \( S \). It has no real roots and \( \text{EulerPhi}[N/2] \) complex roots - but each complex root adds 2 to the dimension, so \( \text{NumberFieldSignature}[z] \) will be the same as \( \text{GenScale} \) with the real and complex roots reversed. In fact \( Q(z) \) is a simple quadratic extension of \( S^* \)

here the output will be simply \( \{3,0\} \) because there are 3 real roots and no complex roots. The actual polynomial is given below.

\[
\text{MinimalPolynomial}[\text{GenScale},x] \\
(* \text{The degree of this polynomial must match the NumberFieldSignature above. The coefficient of the highest term will be 1 so the minimum polynomial of GenScale is 'monic' and therefore GenScale is an algebraic integer and its inverse is also an algebraic integer so GenScale is a unit. If the base field was the integers instead of the rationals, GenScale would be an actual integer (and a unit), so it would be 1 or -1. That would be pretty boring.} *) \\
1 - 9 x - x^2 + x^3
\]

\[
\text{MatrixForm}[\text{Table}[\text{AlgebraicIntegerQ}[\text{scale}[k]],\{k,1,3\}]]
\]

(* \text{GenScale is an example of a 'primitive' scale since it is scale[HalfN] or scale[HalfN-1] if N is twice-odd. Therefore it is a scale[k] where GCD[k,N] = 1. The Scaling Field Lemma shows that All such scales are algebraic integers and also units. It is a non-trivial fact that such scales are also independent over the rationals, so just EulerPhi[N]/2 of them suffice as a basis for \( S^* \))

Here all are primitive so all are integers and in fact units, so all will be True.

\[
\text{AlgebraicNumberPolynomial}[\text{ToNumberField}[\text{hDS}[1],\text{GenScale}],x]
\]

(* \text{The First Family Scaling Lemma says that hDS[1]/hN is always GenerationScale[N]. Recall that GenerationScale[N] is exactly GenScale as long as N is not twice-odd, so in these cases the answer will just be x and DS[1] will be a 'second generation' N. But when N is twice-odd GenerationScale[N] is no longer equal to GenScale, so the answer will be a non-trivial polynomial. But we know what polynomial it will be. It will be the one that converts GenerationScale[N] to GenScale - namely AlgebraicNumberPolynomial[ToNumberField[GenerationScale[npoints],GenScale],x]*)

Therefore the answer here will be \( x \).
Introduction

The truncated extended edges of a regular N-gon define nested star polygons as shown in (i) below for N = 14. These star polygons have a natural scaling based on the horizontal displacement of the intersection (‘star’) points. It is easy to see that this ‘star-point’ scaling is a basis for $\mathbb{Q}_N^+$ - the maximal real subfield of the cyclotomic field $\mathbb{Q}_N$. This subfield has order $\phi(N)/2$ where $\phi$ is the Euler totient function, and it can be generated by $2\cos(2\pi/N)$ - so it defines the (real) coordinates of N. Therefore the star-point scaling can be regarded as a natural scaling of N itself and we call $\mathbb{Q}_N^+$ the ‘scaling field’ of N- which we write as $S_N$.

This scaling is consistent with a ‘family’ of regular polygons (ii), which arise naturally from the constraints of the star polygons. This is what we call the First Family of N. (Note that the first Family for N= 14 contains N = 7 as the central ‘tile’.) This implies that the geometric and algebraic properties of these First Families are inherent in the star polygons – and hence inherent in the underlying polygon N. Every star[k] point defines an associated scale[k] and also an S[k] tile of the First Family.

The last step in (iii) shows the singularity set W (web) which is obtained by iterating the star polygon edges under a piecewise isometry such as the outer-billiards map $\tau$. Therefore W consists of the points where $\tau^k$ is not defined for some k and this makes it a valuable tool for analysis of the dynamics of $\tau$. The first few iterations of W for N = 7 and N = 14 are shown below.

Definition (The outer-billiards singularity set (web) of a convex polygon P)
Let $W_0 = \bigcup E_j$ where the $E_j$ are the (open) extended edges of P. The level-k (forward) web is $W_k = \bigcup_{j=0}^k \tau^{-j}(W_0)$ and the level-k (inverse) web is $W^i_k = \bigcup_{j=0}^k \tau^j(W_0)$. The limiting webs are $W = \lim_{k \to \infty} W_k$ and $W^i = \lim_{k \to \infty} W^i_k$. 
**Example** (The star polygon webs of \( N = 7 \) and \( N = 14 \)) For a regular polygon \( N \), \( \tau^{-1} \) is \( \tau \) applied to a horizontal reflection of \( N \), so the \( W_k \) can be generated by mapping the forward extended edges of \( N \) under \( \tau \) - and if desired a reflection gives \( W_k \) also. Here we choose the level-0 set \( W_0 \) to be the edges of the (maximal) star polygon of \( N \). Below are \( W_k \) for \( k = 0, 1, 2 \) and 10 for \( N = 7 \) and \( N = 14 \). These are what we call ‘generalized’ star polygons.

It is easy to show that these ‘inner-star’ regions are invariant. In [VS], the authors give evidence for the fact that this region bounded by \( D \) tiles, can serve as a ‘template’ for the global dynamics. When \( N \) is even, symmetry allows this template to be reduced to half of the magenta rhombus shown above. This will be our default region of interest for \( N \) even, and for \( N \) odd it will extend from \( D \) to the matching right side \( D \) as shown here for \( N = 7 \).

When \( N \) is twice-odd, the geometry of these two default regions should be equivalent because the equivalence of cyclotomic fields implies that the \( M \) tile of \( N \) can be regarded as \( N/2 \) under a scaling and change of origin. The Scaling Lemma gives the equivalence of scales under scale[\( k \)] of \( N/2 = \text{scale}[2k]/\text{scale}[2] \) of \( N = \text{scale}[2k]/\text{SC}(N) \).

**Lemma 4.2** (Twice-odd Lemma) For \( N \) twice-odd the First Families and webs of \( N \) and \( N/2 \) are related by \( T[x] = \text{TranslationTransform} \{0,0\}-cS[N/2-2][x]/\text{SC}[N\downarrow N/2] \).

Proof: \( T \) maps \( M = S[N/2-2] \) to the origin and then scales it to be a gender change of \( N \) and Lemma 2.3 shows that this is consistent with the First Family Theorem since \( M \) is an odd \( S[k] \). Because \( T \) and \( T^{-1} \) are affine transformations they will commute with the affine transformation \( \tau \) and hence preserve the web \( W \), so \( T[W(N)] = W(N/2) \) and conversely. □
The Web W for \(N = 7\) and \(N = 14\). This web plot would be unchanged not matter which is at the origin (but the dynamics would differ). With \(N = 7\) at the origin, \(hM[1]/hN = \tan(\pi/7) \cdot \tan(\pi/14) = \text{GenScale}[7]\) and this is the scale of the 2nd generation.

This 2nd generation at GenStar or \(\text{star}[1]\) of \(N=14\) is clearly not self-similar to the 1st, but the even and odd generations appear to be self-similar and this dichotomy seems to be common for all N-gons with extended family structure at GenStar (including \(N=5\) above where the periods are high and low). Most cases are similar to \(N=13\) where the even and odd generations at GenStar are related in an imperfect fashion. This dichotomy is apparently driven by the fact that relative to \(D[k]\), these even and odd generations alternate right and left sides. For \(N=7\), the \(M[k]\) transition from odd to even has limiting temporal ratio 8 as illustrated earlier, while the even to odd transition appears to be 25.

\(N = 7\) and \(N = 9\) are ‘cubic’ polygons so they have a second non-trivial primitive scale along with \(\text{GenScale}[N]\). In both cases this competing scale is \(\text{scale}[2] = hS[1]/hS[2]\). These scales are independent and this is consistent with the fact that the First Families of \(S[1]\) and \(S[2]\) have very little in common. We have chosen to use \(\text{GenScale}[N]\) as generator of the scaling field, but any non-trivial primitive scale could play this role. However we showed in Section 3 of \(H[5]\) that \(\text{GenScale}[N]\) arises naturally from the constraints of the star polygons so it seems to play a special role under \(\tau\). For example with \(N = 7\), both \(S[2]\) and \(S[1]\) generate ‘families’ of related tiles in the web \(W\), but once an \(S[2]\) family is generated, it tends to evolve by \(\text{GenScale}\) rather than \(\text{scale}[2]\). This occurs for example in the (very complex) convergence at \(\text{star}[1]\) of \(N = 7\).

Below is the 2nd generation region in vector form. These even generations are called ‘Portal’ generations because the normal \(S[2]\) tiles are replaced by smaller Portal\(M[2]\) tiles which are only weakly conforming to \(D[1]\) and \(M[1]\). These tiles do not exist in the first generation, so there are no \(PM[1]\) tiles. On the right-side of \(D[1]\) there are chains of \(D[k]\) tiles converging to \(\text{star}[3]\) but they alternate real and ‘virtual’. The orthogonal chain on the right consists of odd \(DS3[k]\) and even \(PM[k]\), so the local geometry of \(\text{star}[3]\) of \(D[1]\) is very different from the left-side \(\text{star}[1]\) of \(D[1]\) which has ‘normal’ 3rd generation geometry identical to GenStar.
Below is the web for the second generation at star[1] of N = 14. This is a reflected view from the plot above. This web is based on the dual-center map explained in the Appendix of [H5].

**Example A2** (The edge geometry of N = 14) F[z] = Exp[-Iw][z - Sign[Im[z]]]; w = 2Pi/14; H = Table[{x, {x, -1, -.5, .001}}; Web = Table[NestList[F, H[[k]], 1000], {k, 1, Length[H]}]; Graphics[{AbsolutePointSize[1.0], Blue, Point[{Re[#], Im[#]} & /@ Web]}]

The scaling fields S7 and S14 are generated by x = GenScale[7] = Tan[π/7]·Tan[π/14]. Inside N = 14, S[5] is the surrogate N = 7, so by convention all heptagons are scaled relative to S[5]. M[1] (a.k.a. M1) is a 2nd generation N = 7 but the edge dynamics are much simpler than N = 7 because S[2] is missing. M[2] is the ‘matriarch’ of the 3rd generation and has the same edge geometry as N = 7. The PM[2] are only weakly conforming to D[1] but it is easy to find their parameters. The corresponding PM[1] tiles do not exist. See [H8] for a derivation of PM[2].

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<tbody>
<tr>
<td>x</td>
<td>x</td>
<td>$\frac{1}{2} - 4x - \frac{3x^2}{2}$</td>
<td>$-\frac{3}{8} + \frac{17x}{4} + \frac{9x^2}{8}$</td>
<td>$\frac{3}{7} + \frac{3x}{7} - \frac{x^2}{7}$</td>
<td>x(^2)</td>
<td>x(^2)</td>
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**Evolution of the web**

The large-scale web for any regular polygon is dominated by rings of large resonances called D’s. These ‘necklaces’ of D’s guarantee that the region between rings is invariant and this guarantees that no orbits are unbounded. There is a natural conjugacy relating the dynamics of each inter-ring region. The region inside the first ring is called the central 'star’ region and it serves as a template for the global dynamics. Below are typical webs for N odd and N even. In the N-even case the generating polygon is itself a D and the local geometry of all the D’s is conjugate.

Below are the first 4 iterations of the inverse web for N = 7 in blue with the level 0 forward web shown in magenta for reference. The evolution is similar to N = 5. The central ‘star’ region will be formed from L1 and L2 and their symmetric counterparts. The L3 rays will form the outer edges of the ring of seven D’s and the remaining large scale web.

The local evolution of the canonical D is described below.

To track the web development of D’s external edges, start with the end point of L3 on the extended forward edge and recursively ‘slide and rotate’ as each iteration acts on the previous. Translations and rotation are the key elements of all piecewise isometries acting on polygons.
This web algorithm is a two-dimensional version of the ‘swap domain and range’ cobwebbing used for functions of one variable. Each swap involves a shear in opposite directions along two edges so it is rigid rotation.

The shear for $\tau$ is always the edge length of $N = 7$, which we call $s$. This guarantees that $D$ inherits the same edge length as $N = 7$. The rotation angle depends on the region. The region for $D$ is bounded by the two magenta trailing edges. The rotation angle is $\theta$ - which is half the exterior angle of $N = 7$, so $D$ is a regular 14-gon. When $N$ is even the shear is the same, but $\theta$ matches the exterior angle of $N$, so $D$ is a clone of $N$.

The inner edges of $D$ are generated using the symmetric point on the top edge of this region. Now the shear direction is reversed, so the $L_2$ segments generate the inner edges of $D$ and also define the outer edges of $S_2$. ($L_1$ defines $S_1$ and the inner edges of $S_2$.) The 4th iteration of $L_2$ is aligned with the magenta trailing edge of $N = 7$, and the evolution of $D$’s inner edges will cease. (The edge numbered 4 is shown in magenta here, but it is also blue.)

This web evolution in the vicinity of $D$ is generic for all regular polygons because the $D$’s always have one edge which lies on a trailing edge of the generation polygon and one edge which lies on a forward edge. For $N$ odd, it only takes $N-3$ iterations for the trailing edge to map to a forward edge so the inner edges of $D$ no longer evolve. This implies that the inner star region is invariant after $N-3$ iterations for $N$ odd ($N/2-2$ iterations for $N$ even).

$D$’s region spans 3 forward edges since it is step -3. This span is maximal for $N = 7$ so $D$ has the largest possible measure. The development of $S_1$ and $S_2$ are similar, but complicated by the congestion of the ‘inner star’ region. In all cases the shear is the same, so the centers of these regions are displaced by $s/2$ outwards from the corresponding ‘star’ point -where the forward edges meet the trailing edges. These points are analogs of hyperbolic fixed points.

The rotation angles of these domain regions are the ‘star’ angles. When $N$ is odd, they are of the form $\pi-k\phi$ for consecutive integers $k$, where $\phi = 2\pi/N$ is the exterior angle of $N$. When $N$ is even they are of the form $k\phi$. For example with $N = 7$, $\theta$ is clearly $\pi-3\phi$, and the next two are $\pi-2\phi$, followed by $\pi-\phi$ which is the interior angle of $N = 7$. Therefore $N = 7$ can be generated by these algorithms. Eventually $D$ will have a canonical tile in each of his six step regions. $S_2$ is the only tile which is shared by both $D$ and $N = 7$. For $N = 14$, the shared tile would be a scaled copy of $N = 7$ and in this case $D$ and $N= 14$ would have identical webs and conjugate dynamics.
Example (The web evolution of N = 14). Note the mutations in the cyan odd S[k]. In each case the result is a magenta heptagon with the same center and height as in Lemma 2.3.

The (clockwise) outer-billiards map $\tau$ is defined for ‘most’ points on the forward edges of the N-gon - but it is discontinuous at the star points. This is a ‘shear’ discontinuity of horizontal magnitude $sN$ as the ‘target’ vertex changes abruptly. On each iteration of the web this shear is followed by a position-dependent rotation because every edge of N is being iterated and the 14 domains map to each other under $\tau$.

This ‘shear and rotate’ scenario is common for any piecewise isometry acting on a polygon. See for example the dual-center map in the Appendix. In the example above, the 14 domains are partitioned by the star polygon edges and for N even, the rotation angles are the ‘star-angles’ $k\phi$ where $\phi = 2\pi/N$. (Note that this shear and rotation is consistent with the $sN/2$ center displacement of the S[k].)

D is constructed by a repetition of this ‘shear and rotate’ process. Since the rotation angle is $\phi$, D is a clone of N. D is the only tile formed in a ‘normal’ step-1 fashion. In general S[k] will have edges formed by rotations of step N/2-k, because the numbering of the S[k] is ‘retrograde’ relative to the increase of the star angles.

By symmetry, the S[k] are formed from two competing shears, one relative to N and the other relative to D. These shears are $\phi$ apart because they occur on consecutive edges of N.

For N twice-odd as shown here, it takes an even number of iterations to go from the bottom edge to the top shear. For S[k] with k odd, each step is $j\phi$ with j even, so the top shear and bottom shear are a perfect match and either one would form S[k] independently - so the S[k] are N/2-gons. This can be regarded as a ‘mutation’ or ‘resonance’ relative to the even S[k].

This step-2 evolution of the M tile would have to be consistent with the local First Family of N/2, and indeed this family is always step-2 with respect to D and N as shown in the First Family Theorem. For S[4] shown here, the step-size is 3, so the edge numbering will be 0,3,6,9,etc and this will be period 14 with no resonance with the top sequence, so S[4] will be a 14-gon, but formed in a redundant step-3 fashion. As with S[5], this will have an effect on the local ‘in-situ’ family of S[4]. Note that with S[5] (N = 7) at the origin, this S[4] tile would be called S[2] but it will still have a step-3 relationship with N = 7, so the webs of N = 7 and N = 14 are compatible – but they are only identical in the limit.
For $N$ twice-even the star angle and shears are unchanged but now it takes an odd number of steps to go from the bottom edge to the top shear, so there are no resonances as in the twice-odd case. However all the $S[k]$ (except $D$) will still be formed in the same multi-step fashion, and once again this will influence their ‘in-situ’ families.

**Generation Evolution at GenStar**

Since $D[1]$ exists at GenStar there is hope for a continuation of this symmetry as shown in the enlargement below – where we have omitted $M[1]$. This $D[2]$ will exist iff there is a matching $D[2]$ in the star[1] region of $D[1]$, so $D[1]$ is assuming the role of $D[0]$ and fostering a next-generation $D[2]$. However the self-similarity between $D[k]$ and $D[k+1]$ always alternates between right and left side, so the dynamics of even generations tend to differ from odd generations. There is even a slight ‘bias’ for $N = 5$ where the even and odd generations appear to be exactly self-similar.

Below are the periods for the first 8 families converging to GenStar.

<table>
<thead>
<tr>
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<th>Period</th>
<th>Ratios</th>
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<th>Period</th>
<th>Ratios</th>
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If 7 was a 4k+1 prime, the ratio of these periods would approach N + 1, but here there are two ratios which seem to be 8 (N+1) and 25, so the ratio for even and odd generations appears to approach 200. The M periods for generation k are dependent on the D’s periods for generation k-1, because the canonical M[k]'s are on the edges of the D[k-1]'s. The exact nature of this dependency is not known.


Since the star points of all regular N-gons are transverse intersections of forward and trailing edges, they are candidates for convergent sequence of tiles. It is possible that all invariant measures arise from sequences of tiles converging to primary and secondary star points. The canonical convergence at GenStar is just one of an infinite number of possible sequences. The vector plot above shows a sequence of D[k]'s converging to the star[3] point of D[1] – while from the other direction, a sequence of PM’s and DS3’s converge to the same point.

The D[k] sequence alternates real and virtual, so D[2] is virtual. It is shown in magenta in the vector plot above – along with its reflection. Both of these virtual tiles play a part in the dynamics. Below is an enlargement of this region. It is normal for rings of M’s to form around D tiles and on the right-side of D[1] the rings are centered on a reflection of the virtual D[2]. The blue M[2]'s in this web will be real but truncated to match the colored image shown here. (The extended edges are not unusual for webs.) The result is a string of non-regular pentagons which are formed in a unique fashion. These strings can be seen in the web plots above and below.

On the right side of star[3] (which is also star[2] of DS3), the DS3 chain has physical gaps – which are filled on even generations by PortalM’s. The growth rate of periods is 113 and this matches the growth rate at qstar on the right side of DS3 and also the growth rate at star[2] of N = 7 – which we will investigate in Section 5. In all cases this growth rate skips one generation so
the local Hausdorff dimension is \( \frac{\ln 113}{\ln 1/\text{GenScale}[7]^2} \approx 1.071 \) compared with 1.19978 at GenStar.

The ‘towers’ on the edges of the PM’s below are bounded by the extended edges of the PM’s, but the interior dynamics are determined by pairs of virtual D[2]’s such as those shown on the right. This geometry also exists in the first generation at GenStar. It is called a ‘short family’ because the central ‘M’ is really a DS3 and the rest of the family are DS1 and DS2- which are M[3] and D[3] tiles here. So this is not a traditional M-D relationship and the surrounding magenta webs are different from the traditional webs shown in blue. These magenta webs form inside the virtual D[2]’s. They are 4th generation webs – with embedded M[3]’s surrounded by rings of D[3]’s. This is the first place that such webs appear. This whole region highlights the conflict between D[1] and DS3 – which are step-2 and step-3 of D, so this is the ‘outer-star’ version of the S1-S2 conflict at star[2] of N = 7, which will be discussed in Section 5. (Note that the star[3] convergence shown below takes place on the edge of a tower sitting on top of a PM[2] which is off the screen at the bottom.)

The diagram below shows the evolution of the Portal M’s. They are deformed S1 buds of DS3. The PM’s are smaller than S1 buds, so they can coexist with the local DS3[2]. This determines their dimensions. There is an infinite sequence of PM’s and DS3’s converging to the star3 point of D[1] – which is also the star2 point of DS3. The line of convergence is shown in magenta below. (At mutual star points the two center lines are always perpendicular.)

Following this sequence from the left side, D[2] is virtual and there is strict alternation of real and virtual Dads, which implies that the local self-similarity skips generations just as at GenStar. The DS3’s exist on all generations, but the PM’s alternate generations, so there is no PM[3] here or anywhere else. However there are S2-scaled PM’s which we call PMS2’s.
If the S1 buds of DS3[1] did exist, they would be S2’s of M[1]. Instead M[1] is surrounded by a chain of PMs, along with a lone DS3[2]. Below is an enlargement of the region local to star[3] of D[1], showing the convergence of sequences of PM’s and DS3’s.

We will return to this region below, but first note that that the star[3] point above lies on the top edge of a PM[2] tile and this edge can be used to obtain an exact value for sPM[2]
The side of PM[2] is the distance between the star[1] points of the two magenta M[3]’s. All the tiles in between are canonical, so it is an easy matter to find this distance in the First Generation and then rescale by GenScale^2 to return to the second generation.

Below is what we call the Short Family for N = 7 – because DS3 can be regarded as the ‘matriarch’ with DS2 as the step-2 tile. But this short family structure can only be continued on the left if DS2 is regarded as D[1] and not a step-2 of DS3 – because these two roles are not compatible.

The desired distance is twice the distance shown here between far left and right black markers. The segments above are both differences of star points of known tiles so they can be found in exact form. From left to right:

D1 = star[5][[1]] + star[1][[1]] of DS2 = (s5 + s1)*hDS2 = (Tan[5*Pi/14] + Tan[Pi/14])*hDS2

D2 = star[3][[1]] of DS3 = s3*hDS3 = Tan[3*Pi/7]*hDS3

So the side of PM[2] scaled up to the First Family is sPM = 2h[D1+D2]. (Back in the second generation this is sPM[2] = sPMFF*GenScale^2)

By definition, these heights are:


hDS[2 ] = h(Sin[pi/7]*Tan[pi/7]) and hDS[3 ] = h(Sin[pi/7]*Tan[3pi/14])

so sPM = h(2*GenScale^2*Sin[pi/7]*((Tan[5pi/14]+Tan[pi/14])*Tan[pi/7]+Tan[3pi/14]*Tan[3pi/7])) \approx 0.048267061112875774297

Therefore PMscale = sPM/sN7 = sPM/(2hTan[pi/7] =

Cot[pi/7]*GenScale^2*Sin[pi/7]*((Tan[5pi/14]+Tan[pi/14])*Tan[pi/7]+Tan[3pi/14]*Tan[3pi/7]))

Which simplifies to Sin[pi/7]*Tan[pi/14]^2Tan[pi/7](1 + Tan[pi/14]*Tan[pi/7] + Cot[pi/14]*Tan[3pi/14])
This is clearly in the scaling field $S_7$:

$$\text{AlgebraicNumberPolynomial}[\text{ToNumberField}[\text{PMscale}, \text{GenScale}], x]$$

gives

$$\frac{1}{8} - \frac{3x}{4} + \frac{5x^2}{8} \quad \text{so PMscale} = \frac{1}{8} - \frac{3}{4} \tan\left(\frac{\pi}{14}\right) \tan\left(\frac{2\pi}{7}\right) + \frac{5}{8} \tan\left(\frac{2\pi}{7}\right)^2 \tan\left(\frac{\pi}{7}\right)^3 \approx 0.0501137925752$$

And therefore $s_{\text{PM}} = 2h\tan\left(\frac{\pi}{7}\right) \left[\frac{1}{8} - \frac{3}{4} \tan\left(\frac{\pi}{14}\right) \tan\left(\frac{2\pi}{7}\right) + \frac{5}{8} \tan\left(\frac{2\pi}{7}\right)^2 \tan\left(\frac{\pi}{7}\right)^3\right] \approx 0.0482670611\cdot h$

It makes sense to compare PMscale with the other scales. Our interest is a comparison of PM with $M[1]$ – which is the ‘matriarch’ of this 2nd generation. Since $M[1]$ is scaled by GenScale relative to $N = 7$ we want the ratio of PMscale and GenScale

$$\frac{\text{PMscale}}{\text{GenScale}} = \sin\left(\frac{\pi}{7}\right) \tan\left(\frac{\pi}{14}\right) (1 + \tan\left(\frac{\pi}{14}\right) \tan\left(\frac{2\pi}{7}\right) + \cot\left(\frac{\pi}{14}\right) \tan\left(\frac{3\pi}{14}\right)) \approx 0.455926999989$$

Of course $s_{\text{PM}}/s_{M[1]}$ is this same ratio so PM[2] is less than half the size of M[1].

These PM tiles define a new scale for $N = 7$ and it is likely that there are an infinite number of such scales defined by quadratic polynomials in GenScale. Since PM[2] is canonical, $PM[2] \cdot \text{GenScale}^k$ is also canonical for all $k$ and these PM tiles seem to exist at least for even $k$. There are also ‘S2-scaled’ PM tiles with side (or height) scaled by scale[2] relative to the normal PM’s. We call these PMS2 tiles. (The ‘main-line’ scaling is scale[3] (GenScale) – which is also known as S1 scaling.) These PMS2 tiles occur naturally because any regular tile such as S2 can support a local First Family and these families are identical to the ‘main-line’ families except for the extra scale[2] factor. Of course this means there are PM(S2)$^k$ tiles for any $k$.

**Example:** Below is a portion of a 4th generation S2 family with ‘matriarch’ MS2[3]. The associated PMS2[4] has side $s_{\text{PM}[2]} \cdot \text{GenScale}^2 / \text{scale}[2] = 0.00151843\cdot h$. 
Below is a graphic from earlier showing a ‘normal’ PM[4] with radius rPM[2]-GenScale². This is the second tiles in what we believe is an infinite chain of PM[2k] tiles converging in a self-similar fashion to the star[3] point of D[1]. For odd generations, these PM tiles alternate with DS3[k] tiles. Converging on the left are D[k] tiles alternating real and virtual (so the small D[4] shown below is virtual). As indicated earlier, all of these tiles exist in the top edge of a PM[2]. This type of alternation is not unusual for N = 7.

The shaded region is enlarged below in a vector plot – showing how the PM[4]’s are integrated into the 4th generation in place of S[2]’s of M[3]. This region shows clearly the dichotomy between the ‘normal’ (blue) dynamics of the 4th generation and the unpredictable (magenta) dynamics across the border.

On the left above, note the proximity of D[4] and PM[4]. This is normal for an ‘even’ (Portal) generation. The problem is that D[4] is a ‘main-line’ tile with S1 (GenScale) scaling and this scaling is incompatible with PM scaling. Therefore the dynamics on the right side of D[4] are unpredictable while the region on the left of D[4] is a typical 5th generation with ‘matriarch’ M[4]. The self-similar ‘towers’ on the edges of PM[4] are a testament to the clash in scaling between D[4] and PM[4]. The situation is much worse when S2 scaling is introduced into the mix, as shown below. Note that the proximity of D[4] and PM[4] is unchanged from above, but now they are surrounded by S2[4] tiles – which are larger and more dominant because hS2[4]/hD[4] = 1/scale[2] ≈ 2.60388.
**Inner star dynamics vs. outer star dynamics**

The dynamics of the inner star region are dominated by the interaction of the S1 and S2 buds and hence by the interaction of the corresponding scales. The outer star region is dominated by S1(GenScale) dynamics. This can be seen in the 2nd generation where S2[2] is missing and this allows S1[2] to play the role of a D[2] and raise a family on the edges of M[1]. By contrast in the inner star region, S1 cannot support a ‘normal’ family. Instead his edges support a hybrid family which is a mixture of S1 and S2 scales.

However the differences between inner star and outer star dynamics are scale dependent. Self-similarity demands that that at some scale difference, the outer star and inner star must have the same dynamics. For N = 7 there is a 2 generation scale difference, so there is an S2[3] in the vicinity of M[1] with the ‘same’ dynamics as S2[1] at M. In fact the small scale dynamics at star1 of M[1] are identical to the dynamics at star1 of M[0], but scaled by GenScale^2. We will demonstrate this below, starting with star1 of M[0].

The three regions outlined below share the same small-scale dynamics. The left-side dynamics at star[2] are identical to the S2 dynamics at qstar, and the right side is identical to the top of S1 so we will concentrate on the top of S1 and star[1].
The enlargement below shows that S1 is playing the role of a D[1] and fostering a M[2] on each edge along with the corresponding D[2]'s at each vertex. These seem like ‘normal’ families, but on closer inspection they are far from normal.

The central M[2] has radius GenScale^2 so she should be the matriarch of a 3rd generation which is self-similar to the 1st generation. But this M[2] is living in two different worlds. Vertically she has ‘normal’ 3rd generation buds with matching D[2]'s and a small M[3]. The missing buds in the vertical direction are her two S1's which have dissolved into chaos. We will resolve this region below when we look at star[1] of M[0].

In the horizontal direction, M[2] has the same dynamics as a DS3. Note the Portal M’s on four of her sides in place of S1 buds. This means that the S2 buds on her right and left did not arise as her progeny, but instead they are same-generation DS2 buds with a (virtual) S2[2] D, similar to the one in the upper left corner above. These two S2[3] buds map to each other, but not to the remaining three buds, so in the horizontal direction M[2] behaves like a S2S3[3]. The radius of an S2S3[3] is scale[2]*GenScale^2/scale[2], so she is the same size as a M[2], but dynamically she is related to the S2 family.

Note that this virtual S2[3] and the real D[2] share the same vertex at the local GenStar point, and this virtual S2[3] shelters a perfect family with patriarch S2[4]. Except for the S2 size difference, this family is identical to a normal 5th generation family. There is an infinite string of S2 D’s and S2M’s converging to the local GenStar point and the ratio of periods is the same as the ratio found at the traditional GenStar point.

The new S2[4] ‘D’ tile can be seen below. On his left is a MS2[4] and the rest of the family. Given adequate space, it is natural for a M-D pair to generate rings of congruent D’s and M’s as seen here. The first ring of two D[4]’s has normal spacing, but then the ring structure breaks down because it conforms to the virtual S2[3] and not the actual D[2]. This leaves three D[4]’s surrounded by larger S2[4] D’s. The blue and red regions contain dynamics unlike anything seen before. A PM[4] sits on a virtual edge of MS2[3] with a strange mantle. Just like the D[4]’s, this tile is totally out of place in this region. It is not the PM of MS2[3]. If MS2[3] had a PM it would have to be scaled by GenScale/scale[2] relative to S2S3[3], but this one is slightly larger because it is scaled by scale[2].
The region close to MS2[3] has normal Portal generation dynamics with S1[4] buds which inherit the S2 scale from M so they are the same size as the S2[4]'s. The M[4] at the vertex of MS2[3] is also standard for a Portal generation, so the chaotic region can be regarded as the remnants of a PMS2[4].

For comparison, below is a canonical 4th generation S2 family with a PortalMS2[4]. Compared to the original PM[2], her radius would be \( r_{PM[2]} \cdot \text{GenScale}^2 / \text{scale}[2] \approx 0.0017498129 \). Note that there is a virtual edge here, just as above. In both cases, the length of this virtual edge is the same as the side of MS2[3], so it supports a complete 5th generation. However, this 5th generation cannot fit on her side because the D’s are not at the vertices.

The vector diagram below shows two virtual D[4]'s which have the canonical S1,S2 relationship with the DS2[4]'s at top and bottom. The real D[4] in the center does not share this relationship, but he is the central D in a ring of three surrounding a M[4] – which is a step-3 of the S2[4]'s. The virtual MS2[4] shown here is a matching M for the S2[4] on the left. She also has a cozy relationship with the S2[4]'s at top and bottom and she also has a major impact on the dynamics at the top of D[4] as well as the dynamics around PM[4].
Below is a vector plot of the region at the top of D[4] which is marked by the small rectangle above. The M[5] is actually an S2S3[6] so this is locally similar to the top of S1 but three generations removed. Note the identical S2S3[6]’s at top left and right. These are in the correct positions to be step-3 of the large S2[5]’s. Vertex 7 of M[5] is shared with virtual MS2[4] and also virtual MS2[5]. This mimics the Portal generations where the M’s have next-generation M’s at their vertices. Since ‘M[5]’ is really an S2S3[6], the S2[6]’s are normal S2[6] D’s, but there are no MS2[6]’s.

The same region is shown below using orbit plots to fill in the dynamics. Give the unique mixture of S1 and S2 tiles, it is no surprise that the small-scale dynamics are unpredictable, but there are small invariant islands of canonical behavior. One of these is the 8th generation family noted below. There is another equivalent family at the base of D[5] and in general the dynamics at D[5] is very similar to the dynamics local to M[5] – because M[5] is also a step-3 relative to the large S2[5]’s.
As indicated above, the dynamics local to $S2[6]$ and $M[5]$ are very similar to the dynamics local to $D[5]$ – a mixture of canonical behavior (in blue below) and exotic dynamics in black. The blue region is a perfect 8th generation $S2$-scaled First Family presided over by $S2[7]$ and the matching $MS2[7]$. Since this is an even generation, it is dominated by Portal[M’s], but the next generation will no doubt be a perfect 9th generation $S2$-scaled First Family presided over by $S2[8]$ and $MS2[8]$. (The $D[7]$ on the right is 3 generations removed from $D[4]$ – which is the central tile in this region- so it could be a case of ‘deja-vu all over again’. More on this below.)

The left side of $S2[6]$ has dynamics that have never been observed before – but it is typical for such regions to have ‘islands’ of canonical behavior. This region has been studied over a period of years and the plot above is based on trillions of iterations, because the ‘return’ periods are on the order of 2 million. To get an idea of the scale, the radius of $MS2[7]$ is $GenScale[7]/scale[2] \approx 5.07424 \times 10^{-7}$. The canonical tiles embedded in the black region, provide partial checks on the accuracy of the 40-decimal place iterations of orbits that we hope are non-periodic - or at least correspond to tiles which are generations removed. The enlargement below shows some of the embedded canonical tiles. A simple reflection of the blue region about the center of $S2[6]$ is enough to uncover the larger embedded tiles which have survived in the chaos of the black region. There is no doubt that the black region is influenced by ‘virtual’ $S2[7]$ and $MS2[7]$ tiles which we do not show here.

It appears that there is no $S2[8]$ at the local GenStar point on the left side, but there may be a matching $MS2[8]$. If this is true, there may be a return to canonical dynamics at this point. This would mimic the ‘blue’ dynamics on the right. Note that the PM and $M[7]$ pair on the edge of $M[6]$ are canonical. This configuration occurs in the second generation – where the first PM[2] occurs (There is no PM[1].) The existence of this pair on the edge of each $M[6]$ makes clear that
these two are primarily step-3 tiles of S2[6]- and these conflicting roles are a critical factor in the breakdown of dynamics.

The configuration around the mysterious D[7] is enlarged below. This vector plot may need to be revised as new orbits are plotted, but there is a precedent for MS2[7] and the M[7]'s at the vertices. This mimics the M[5]'s at the vertices of virtual MS2[5] at the top of D[4]. However the similarity seems to end there and there is no precedent for the D[7] shown here. This geometry has never been observed before. It is often the case that unique configurations of canonical tiles leads to unique local dynamics and this was certainly true of D[4]. It is reasonable to suspect that the relative geometry of these canonical tiles is what drives the local dynamics – and conversely. In that case the dynamics of this region should be very interesting indeed – but the computational effort to generate plots like this is a real issue.
The plot below is a broader view of the region around D[4]. As usual this is a mixture of vectors and orbits, so we are making some assumptions as to the actual tiles. It is clear that the large virtual MS2[4] plays a major role in the dynamics of this region. The blue region is dominated by a PM[4] which shares some dynamics with the edges of virtual MS2[4]. The PMs only appear on even generations, so the radius is r·GenScale² where r is the radius of PortalM[2] (about .055622). As indicated earlier, this PM[4] is in a region which would normally be dominated by a larger PMS2[4].

The non-regular hexagons on the edges of PM[4] have never been observed before and their position would normally be occupied by an S1S3[5]. These regions are period 32,586 with doubling and this allows us to find the center and vertices with high accuracy. Not surprisingly, the center lies on the line joining the centers of the two matching S2[5] buds. The magenta line of symmetry extends all the way to the local GenStar at the foot of D[2]. Df web scans cannot begin to resolve detail on this scale, but Df orbits could be used. The lower left and upper right corner of this plot are at \{-0.9084, -0.482\} and \{-0.9069, -0.4806\}, relative to N = 7 at the origin with radius 1. (The older plots used radius 1 instead of apothem 1, but there is no issue with conversion between the two conventions.)
The dynamics of the star[1] point of M

If we regard S1 and S2 as D’s, the corresponding M’s would be nested inside of M[0] as shown below. This type of nesting is generic for all regular polygons, so N = 13 has 6 such M’s. These virtual M’s may have real progeny as shown below. In this case the nesting could lead to a sequence of S2 and S1 buds converging to the star[1] point at vertex 5 of M. The first two terms in such a sequence are shown below.

The step-2 dynamics of S2 are naturally tied to the vertices of Mom while the S1 dynamics are more closely related to the edges of M. Therefore it is not surprising that the dynamics at star[1] are largely driven by the dynamics of the S2 family.

Looking at the sequence above, it is easy to imagine an infinite sequence of S2 and S1 buds converging to the star[1] point. This scenario does occur but in a non-trivial fashion. We will trace this evolution here.

The first two terms in the S1,S2 sequence are the self-similar pairs S1[1], S2[1] and S1[2], S2[2] as shown above. There is an S2[3] as shown below which should continue the sequence, but there is no matching S1[3]. The missing S1[3] should be an S1 of the virtual Mom[2] shown below. However this virtual M[2] is playing two different dynamical roles, just like the real M[2] at the top of S1. Here the orientation is rotated by 90°, so there appears to be a normal family in the horizontal direction with a D[2] on the left, but dynamically this M[2] is also a S2S3[3] and the S1 buds are mutated PMS2[4]’s just like vertex 7 of ‘M[2]’ at the top of S1.
Below is an enlargement showing the S2[3] bud and the chaotic region where PortalMS2[4] should be. This is an extended version of the region to the left of MS2[3] at the top of S1. Now there are two PortalM[4]'s instead of one, but the dynamics are identical. The plot below shows how the second ring of D's is formed from extended edges of the S2[3] below. This second ring is not compatible with the first ring and this conflict disrupts the dynamics. The two S2[3] buds are in the correct position to foster an S1 between them and this S1 would fit perfectly on the edge of M[2], so in this sense the chaotic region can be regarded as the debris from a failed S1, but since M[2] is also an S2S3[3], the S1 would not have formed anyway.

At star[1] this represents one complete cycle which spans four generations. The next cycle is initiated by MS2[4] as shown below.

The S1 and S2 buds are S1[5] and S2[5] but their full names are S1S2[5], S2S2[5]. This reflects their dynamical history. A S2S2[5] bud will have radius $rD*GenScale^5/(scale[2])^2$ while any S1S2[5] bud has radius $rD*GenScale^5/scale[2]$. Of course these names do not always reflect the points in their evolution where the S2 scale occurred. These two buds play the part of the original S1[1] and S2[1], so the first cycle extends from S1[1], S2[1] to S1[5], S2[5]. It is important to note that each cycle requires another S2 scale, so S1[5] and S2[5] are actually S1S2[5] and S2S2[5].
There is one important difference between this scenario and the one observed earlier at the top of S1. In both cases, there is a 5th generation family converging to the local GenStar point, but at star[1], the GenStar point is offset, so it is necessary to track the S1,S2 sequence instead of the D-M sequence. Of course there is a simple formula for the S1,S2 centers, just like the D-M centers, but the actual sequence has exceptions where buds fail to exist. This is similar to the exceptions found in the pure S2 sequence. Below is the continuation, up to the 7th generation where S2S2[7] plays the role of the ‘orphan’ S2[3] with no matching S1[7].

Below is the re-emergence of chaos in the 8th generation between S2S2[7] and virtual MS2[6]. This is a clone of the 4th generation diagram from earlier, but now the S2S2[7] bud below is virtual because all this is sitting on top of a (real) MS2[4].
It is almost certain that this sequence continues forever. It is an easy matter to compute the centers of these buds to any desired accuracy because distances scale by the size of the buds and the scale factor for a full cycle is \( \text{GenScale}^4 / \text{scale}[2] \approx 0.0003800739001151979652717427 \). The table below gives the periods for the first few terms. The periods grow very rapidly over 4 generations, so the main-line sequences are very difficult to track even though we know their centers to high accuracy.

<table>
<thead>
<tr>
<th>Sequence number</th>
<th>Type of Buds</th>
<th>Periods</th>
<th>Ratios (approx.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S1[1] S2[1]</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>S1[9] S2[9]</td>
<td>14312711</td>
<td>8594019</td>
</tr>
</tbody>
</table>

The convergence of the ratios seems to follow the traditional ‘high-low’ alternation which is consistent with the real-virtual inversion that takes place after each cycle. This same alternation takes place at GenStar where there is also a real-virtual dichotomy. It is still not clear what the exact limiting value is, but the best guess is 1254. This would make the ratio 1 mod 7 in keeping with the ratio of 8 at star[3] and 113 at star[2].

This is going to be hard to pin down further because the expected period of S1[13] is about 18 billion and the expected period of S2[13] is about 11 billion. Even on the fastest consumer computer, 1 billion iterations of the Tangent map takes about 24 hours and this task is difficult to run in parallel. Over a period of 3 days we ran the S[4], S[8], S[12] sequence and the results are:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th>Ratios: 1256.15 1253.819966</th>
</tr>
</thead>
</table>
The Dynamics at the star[1] point of M[1]

Below is the second generation with matriarch M[1]. The S1[2] buds are normal, but there are no S2[2] buds. Recall that these S2 buds are the failed S1’s of DS[3][1], and once again we will see that chaotic dynamics arise when this is repeated in the 4th generation.

The missing S2 buds allow the S1[2] buds to form rings which are truncated by the dynamics surrounding the Portal M’s. At the star[1] point of M[1], the dynamics are ‘generic’ and mimic the star[1] point of M[0]. However the dynamics are two generations removed because S2[3] now plays the role of S2[1].

In the chain of S1’s and S2’s converging to the star[1] point of M[1], there is a gap of 4 generations between self-similar ‘families’, just as observed at star[1] of M[0]. So S2[7] marks the start of the second cycle. However the new S2[7] is actually an S2S2[7] - so each cycle introduces another S2 generation in the same fashion as star[1] for M[0]. This means that the scale for the chain of S1’s and S2’s is GenScale^4/scale[2]. It appears that the ratio of periods is the same as that of M[0].

Even though the small scale dynamics at M[1] match those at M[0], there are significant differences in the large scale dynamics. In the absence of the large S2[2] buds. M[1] is fostering a 3rd generation family on her edges. The edges are the right length but the D[2]’s should be at the vertices instead of the M[2]’s.
The virtual M[2] generates S2[3] and S1[3] to start the chain, but to continue, it is necessary to use a virtual S2[3]. For M[0] this was not necessary since there was a real S2[1] below the star[1] point. However the end result is the same – just two generations removed. The vertices of S2[4] below mimic the dynamics of the chaotic region from the failed PortalMS2[6].

The large-scale web

For all regular polygons, there are endless concentric rings of D’s which confine the dynamics just like the invariant rings of the KAM Theorem. These rings form slightly irregular 2N-gons. with even and odd edges differing by one D. Inside the first ring of Ds (the star region) all orbits are concatenations of 1-step, 2-step or 3-step. The first 4-step appears only outside the first ring. For example, the first DS3 outside has step sequence (333334) which is period 42. All of the rings can be characterized in a similar fashion. Recall that the first ring of D’s has step sequence
(3). The second ring of D’s has step sequence (334) and then next has (33434) and each successive ring adds another (34) to the step sequence. Therefore the periods of the centers are 7, 21, 35,...

If a tile has a periodic orbit, the points in any image of this tile will have the same step sequence (except for cyclic rotation), but this does not imply that all points will have the same period. On each iteration of \( \tau \), the tiles are inverted, so periods will tend to be even, but it is possible for a center of symmetry to have an odd period. This can only occur if there are an odd number of tiles in the orbit.

Below are iterations 0 – 4 of the inverse web for \( N = 7 \) along with levels 1-2 of the magenta forward web for reference. The enlargement shows level 6 where the canonical D is almost a ‘stable’ tile. The extension of his outer edges are forming a new First Family. The magenta level 2 segment now plays the same role as the earlier level 0 segment – because it defines the bounds of the third iteration just as the level 0 edges define the bounds of the first iteration. All this takes place within the same domain region as the canonical D, so the rotation angle is unchanged. The step-3 between edges of the new D shows that this is a sub-domain of the original D.

The horizontal spacing between D’s is clearly \(|2(\text{GenStar}[[1]] - s/2)| = 8.76257\). The D’s in rings 0,1 and 2 shown here, have periods of 7, 21 and 35, and in general they are odd multiples of 7. None of these rings decompose so the periods of the centers are the number of tiles.
The magenta arcs connect the D centers. These magenta polygons are actually the orbits of the centers under the return map $\tau^2$, so under this map, each D center maps to its neighbor – clockwise here. Ring 0 is the only ring that can be replicated by simple rotation. The remaining rings form non-regular polygons. These non-regular polygons are of interest in their own right and we will study their dynamics below. From the standpoint of stability, the important issue is that these rings exist at arbitrary distances from the origin.

For $N$ odd, the D’s in Ring 0 have constant step sequence $\lfloor N/2 \rfloor$ so their winding number is $\frac{\lfloor N/2 \rfloor}{N}$. This is an upper bound for the points inside this ring (the ‘star’ region). It is not clear whether there are points inside the star region with winding numbers arbitrarily close to D. All points outside of this star region have step sequences composed of just $\lfloor N/2 \rfloor$ and $\lfloor N/2 \rfloor + 1$. Clearly $\lfloor N/2 \rfloor + 1$ is the maximum possible step for a regular polygon with odd number of sides and the limiting horizon step-sequence for $N$ odd is $\{\lfloor N/2 \rfloor, \lfloor N/2 \rfloor + 1\}$.

In any ring, the D’s step sequences serve as bounds for the remaining points – just as they do in the star region. For example with $N = 7$, the upper bound for all points in the interior of the star region is $3/7$. The D’s in Ring 1 have period 21 orbits (mapping centers) and the step sequences are constant $\{3,3,4\}$ so $\omega = 10/21$. This is an upper bound on $\omega$ for all points in this ring. Each new ring adds another $\{3,4\}$ to the step sequence, so the step sequence for the second ring is $\{3,3,4,3,4\}$ which is period 35 with $\omega = 17/35$.

<table>
<thead>
<tr>
<th>Ring</th>
<th>Step Sequence</th>
<th>Period of center</th>
<th>Winding Number - $\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>{3}</td>
<td>7</td>
<td>$\frac{3}{7}$</td>
</tr>
<tr>
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<td>{3,3,4}</td>
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<td>$\frac{10}{21}$</td>
</tr>
<tr>
<td>2</td>
<td>{3,3,4,3,4}</td>
<td>35</td>
<td>$\frac{17}{35}$</td>
</tr>
<tr>
<td>$k$</td>
<td>{3} + $k*{3,4}$</td>
<td>$7(2k+1)$</td>
<td>$\frac{(3(k+1) + 4k)}{7(2k+1)} \rightarrow 1/2$</td>
</tr>
</tbody>
</table>

We saw above the first stages in the web for $N = 7$. Since each ring is invariant, it can be generated independently. Below we have partitioned the $N = 7$ web into segments to match each invariant region and these segments are iterated here with contrasting colors. These are iterations 0, 1, 4, 8, 16 and 200. It would make no difference if all the segments are joined and iterated together. These webs is fairly linear in their development as witnessed by the periods of the D’s. The ratio of D’s periods in consecutive rings approaches 1.
The Pinwheel Map

To study the global properties of the Tangent map for any N-gon, it is helpful to use the return map $\tau^2$ and the related Pinwheel map pioneered by Richard Schwartz. The Pinwheel map filters orbits to lie only on specific ‘strips’. We will illustrate the Pinwheel map for $N = 7$ where there are 7 possible strips. They are shown below superimposed on the first three rings of D’s. The strips define transitions in the dynamics of the return map, $\tau^2$. For $N = 7$, there are only 14 possible displacements given by the vectors $V_1, V_2, \ldots, V_7$ and their negatives.

For example $V_1 = 2*(v_1-v_4)$ where $v_1$ and $v_4$ are the respective vertices of $N = 7$, so $p_2 = p_1 + 2*V_1$ and $p_3 = p_2 + 3*V_2$. These are ‘accelerated’ orbits which can be used to analyze large-scale dynamics. (The point $p_1$ is an off-center point in one of the D’s, so it has period 70, but under the return map it is period 35 as shown here.) Under the Pinwheel map restricted to the primary strip, $p_1$ is a fixed point, so the filter ratio is 35 to 1 but no information is lost.

Recall that the winding number of each D in Ring 2 is $17/35$ and for off-center points this translates into $34/70$. These 34 steps are shown here. At a ‘reasonable’ distance from the origin (after Ring 1) any point in the interior of Ring 2 must visit each of the 14 regions on each transversal. In each region, the count is less than or equal to the count for Ring 2, so it can never exceed 34. This is an example of a ‘winding number’ trap - similar to the one defined by the first ring of D’s – the inner ‘star’ region.

The green orbit is typical. Only the first transversal is shown here. It visits each region at least once before returning to the primary strip. The total number of steps is 30. This orbit is period 182 so the return orbit is period 91 and the winding number is $616/(7*182)$. 
Projections.

N = 7 has 3 projections which we call P1, P2 and P3. The P1 projection is just the $\tau^2$ orbit of the point. This projection does not remap any of the vertices. It is shown on the left below. The P2 and P3 projections are mod-2 and mod-3 re-mappings of the vertices. The geometry of these projections can be a valuable aid in analysis of orbits. See Projections.

**Example 1**: \( q_1 = cM[1] \approx \{-3.5135187892997068, -0.801937735804838252\} \). This point is period 28 so the projections are period 14.

\[ \text{Ind} = \text{IND}[q_1, 14] = \{7, 3, 6, 2, 4, 7, 3, 6, 14, 7, 3, 5, \ldots\} \] These are the corners visited by the original orbit so \( \text{PIM}[q_1, 7, 1] \), \( \text{PIM}[q_1, 7, 2] \) and \( \text{PIM}[q_1, 7, 3] \) will take these in pairs and generate the three projections. We usually plot P1 as points and the remaining projections as lines. The 'poly' command below is the same as the 'Line' command in Mathematica, but it joins the first and last point.

\( k = 14; \)

**GraphicsGrid**[{Graphics[{poly[M], AbsolutePointSize[3.0], Blue, Point[PIM[q1,k,1]]}], Graphics[{poly[M], Blue, AbsolutePointSize[1.0], poly[PIM[q1,k,2]]}], Graphics[{poly[M], Blue, poly[PIM[q1,k,3]]}]}, Frame -> All]
Example 2: \( cM[2] = CFR[GenScale^2] = (1 - GenScale^2) \times GenStar \approx \{-3.89971164872729442807252673198, -0.890083735825257617155610257987\} \) Period 98 so the projections have period 49.

\[ k = 49; \]

As expected, the even and odd M’s have different projections.

Example 3: \( cM[3] \approx \{-3.9421605250865362957195, -0.899772414849448879516633\} \) Period 2212

\[ k = 1106; \]
Below is a plot of the ‘chaotic’ region on the top of S1. We will choose an initial point in this region and generate the projections. The point q shown below has coordinates \{-0.91016767268142,-0.4814274969508659\}. We will generate the first 1 million points in the orbit and use these to generate the first 500,000 points in the projections.

\[
\text{Ind} = \text{IND}[q,1000000] = \{6,7,1,2,3,5,7,2,4,6,\ldots\} \quad P1 = \text{PIM}[q1,500000,1]; \quad P2 = \text{PIM}[q1,500000,2]; \quad P3 = \text{PIM}[q1,500000,3]
\]

These projections are tightly folded but they can be filtered by restricting the points to any region of interest. To do this, first generate the full orbit as shown above and then find the indices of the cropped points. In the example below we will restrict the orbit to the first generation so the new P1 points are shown on the left below. On the right is the beginning of the new P2 in magenta compared to the original P2 in black. The ratio is about 1 to 5. Further filtering can reveal structure at small scales.
Generation Filtering

The example above showed an orbit filtered to the first generation. For 4k+1 prime polygons and for N = 7 there are well-defined future generations so it is possible to filter each subsequent generation. Starting with a periodic orbit such as a M[k] or D[k] and filtering it by generations, the cropped points retain the symmetry of the periodic orbit and the individual projections are locally periodic. Here is an example:

**Example:** The filtered P2 projections of M[4]

By definition, M[4] is matriarch of the 5th generation at Gen Star so this tile is a fixed point in the 5th generation as shown on the left below. In the 4th generation it has a (return) orbit of period 3 and in the 3rd generation the period is 13. Subsequent generations yield periods of 205 and 2115. The last plot on the right below is the unfiltered P2 projection. It has period 8743, so the traditional period is $8742 \cdot 2 = 17484$. These projections can be quite large. M[0] is shown for comparison. Click on any image to enlarge.

The P1 projection of an orbit is the ordinary orbit under $\tau^2$. These orbits are typically plotted as points instead of vectors joining successive points. Below are the P1 point orbits for generations 5, 4 and 3. As above, the local periods are 1, 3 and 13.

When we carry this out for the remaining even generation M’s, these same values reaccur – which supports the view that the generations are self-similar. In odd generations, the sequence above is replaced with 1, 4, 54, 256, … as shown in the table below.

<table>
<thead>
<tr>
<th>Period of</th>
<th>Projections of M centers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gen0</td>
<td>Gen1</td>
</tr>
<tr>
<td>M[0]</td>
<td>N/A</td>
</tr>
<tr>
<td>M[1]</td>
<td>14</td>
</tr>
<tr>
<td>M[2]</td>
<td>49</td>
</tr>
<tr>
<td>M[3]</td>
<td>1106</td>
</tr>
<tr>
<td>M[4]</td>
<td>8743</td>
</tr>
<tr>
<td>M[5]</td>
<td>216734</td>
</tr>
<tr>
<td>M[6]</td>
<td>1741397</td>
</tr>
<tr>
<td>M[7]</td>
<td>43319962</td>
</tr>
<tr>
<td>M[8]</td>
<td>348263951</td>
</tr>
</tbody>
</table>
Returning to M[4], it is an easy matter to crop the indices of the period 8743 return orbit to any subsequent generation. For example the 13 Generation 3 points shown above have indices: 1, 166, 1171, 2341, 2506, 3511, 4846, 5234, 6239, 6404, 7574, 8579, 8744 (=1). The periodicity becomes apparent by taking first differences: \{165, 1005, 1170, 165, 1005, 388, 947, 388, 1005, 165, 1170, 1005, 165\}.

Below is a shorter table showing the geometry of the major projections.

<table>
<thead>
<tr>
<th>P2 Projections of M centers - and their periods</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>M[1]</td>
</tr>
</tbody>
</table>
For a given M[k], each projection provides the scaffolding for the next. These projections are ‘uniform’ subsets of each other so the overall scale is unchanged between generations. For example below is the blue period 13 orbit of M in generation 3 which provides the scaffolding for the period 205 orbit from generation 2.

These tables point toward a structure that may allow us to fully analyze the evolution of the M’s. This would be an important step toward analyzing the global dynamics. It seems that every M[k] follows the same periodic evolution which depends only on whether k is even or odd. Using each generation as a 'filter' they all have the 'same' dynamics. This could yield a proof of the self-similarity for even and odd generations, but more importantly it may be a tool for analyzing other polygons. Of course generation filtering can only be used for polygons with well-defined generation structure. This should include all regular 4k+1 prime polygons and maybe a few more 'strays' like N = 7 and some of the composites.

There are two ways to get the entries in the table. For example to get the M[7] Gen4 entry, we could track the orbit of M[7] for a few million iterations and crop the P1 points to Gen4. This would yield cropped points with period 4874. Or we could back up to Gen2 and generate 100 million points (5 or 6 hours) to get P1 points with period 969110, and then cropping these (a few seconds) would give all the other periods. The P2 and P3 projections would be 'free' as long as we kept track of the indices.
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Links
(i) The author’s web site at DynamicsOfPolygons.org is devoted to the outer billiards map and related maps from the perspective of a non-professional.

(ii) A Mathematica notebook called FirstFamily.nb will generate the First Family and related star polygons for any regular polygon. It is also a full-fledged outer billiards notebook which works for all regular polygons. This notebook includes the Digital Filter map (which is only applicable for N even). The default height is 1 - to make it compatible with the Digital Filter map. For investigations without the Digital Filter map, it may be preferable to use one of the notebooks described below – with the more natural convention of radius 1.

(iii) Outer Billiards notebooks for all convex polygons (radius 1 convention for regular cases). There are four cases: Nodd, NTwiceOdd, NTwiceEven and Nonregular.

(iv) For someone willing to download the free Mathematica CDF reader there are many ‘manipulates’ that are available at the Wolfram Demonstrations site - including some outer billiards plots based on the author’s results in [H2]. At the author’s site there are cdf manipulates at Manipulates – which can be downloaded. (The on-line versions have been phased out by most web browsers for security reasons.)

(v) The open source PARI software at pari.math.u-bordeaux.fr has impressive facilities for computer algebra and algebraic number theory.