GeneticsOfPolygons.org

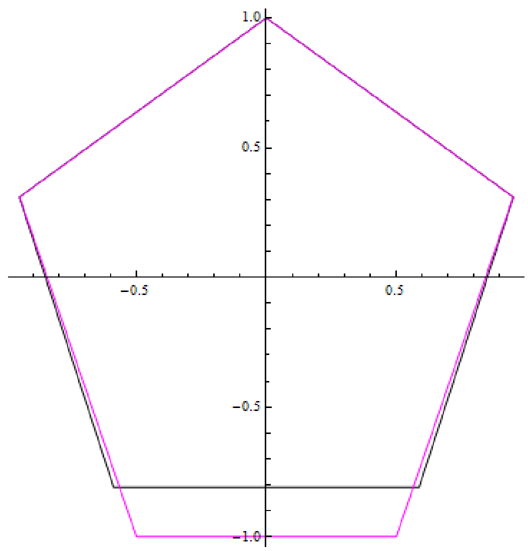
# Summary of dynamics of a non-regular pentagon: N5Kite

The coordinates of the vertices are obtained by rationalizing vertices 1, 3 & 4 of the regular pentagon and preserving y-axis symmetry.

$\mathbf{c_1 = \{0, 1\}; c_2 = \{.95, -\frac{1}{4} + \frac{\sqrt{5}}{4}\}}$ ; $\mathbf{c_3 = \{\frac{1}{2}, -1\}; c_4 = \{-\frac{1}{2}, -1\}; c_5 = \{-.95, -\frac{1}{4} + \frac{\sqrt{5}}{4}\}}$

**N5Kite = {c1, c2, c3, c4, c5};**

Below is the regular pentagon in black overlaid with the N5Kite in magenta




On this scale it looks like vertices 2 and 5 are the same as the regular case, but they are slightly different. We have retained the original y coordinates of these vertices, but the x coordinates have been rationalized to .95 and -.95 instead of

$$Sin\left[\frac{2\pi}{5}\right] = \sqrt{\frac{5}{8} + \frac{\sqrt{5}}{8}} \approx 0.951056516295153572116439333379$$

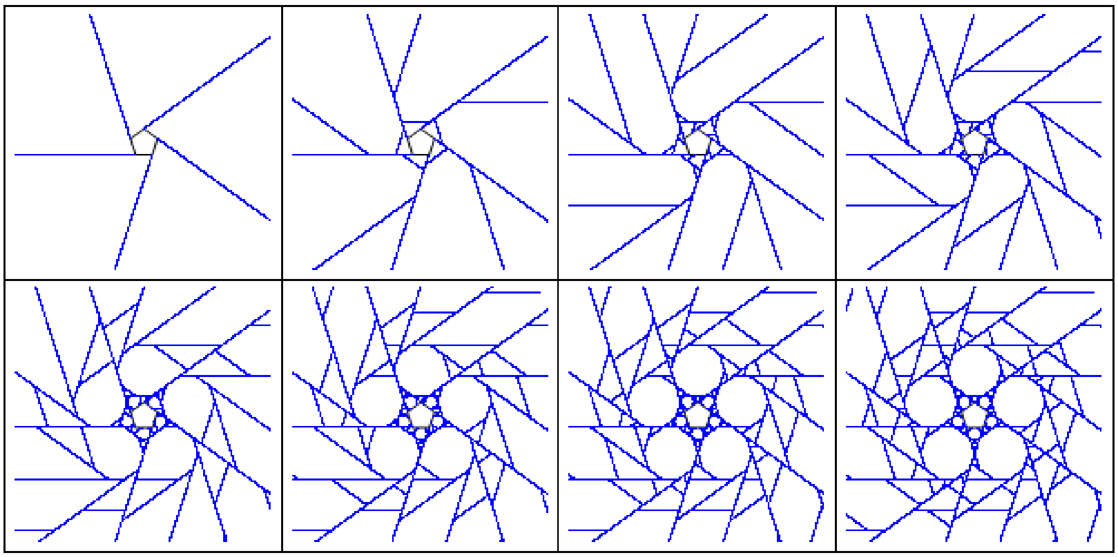
The purpose of this rationalization is to obtain a polygon which is 'close' to a lattice polygon but with just one coordinate irrational. (In this case two symmetric y coordinates are irrational.) The dynamics of a polygon change radically when passing from lattice polygons to irrational polygons. No lattice polygon can have unbounded orbits, but numerical evidence indicates that such orbits are easy to find for most irrational polygons. We will give some numerical evidence below for the N5Kite. The difficult part is to prove that any such orbit is indeed unbounded. Based on the results of Richard Schwartz for the irrational Penrose kite, we call this polygon the N5kite.

We will begin by comparing webs for the regular case and the N5kite - first on a large scale and then on a smaller scale:

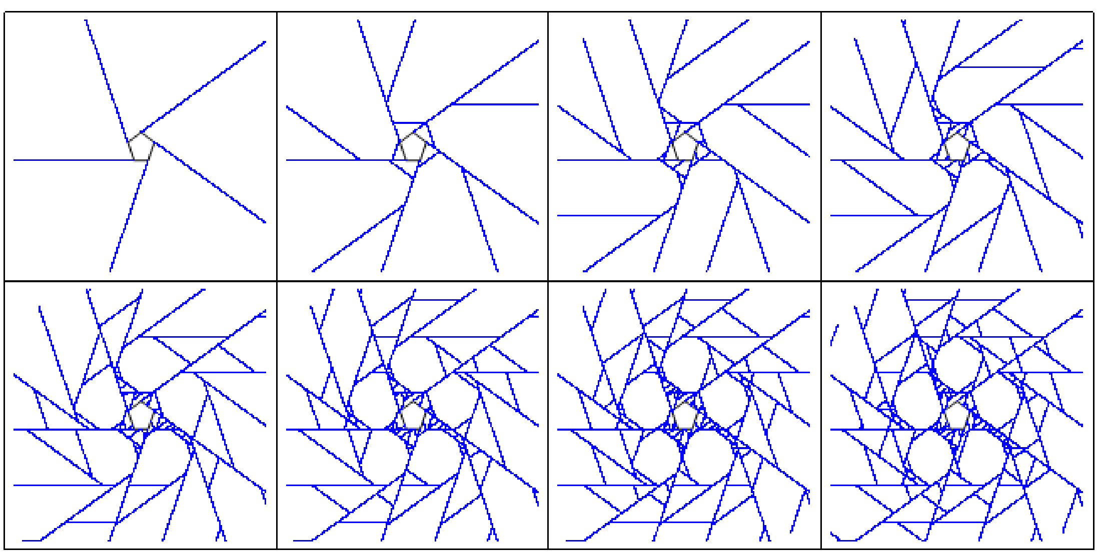
**Gr[depth_] := Show[Graphics[{AbsolutePointSize[1.0], poly[Mom], Blue, Point[WebPoints[.01, 12, depth]]}, PlotRange -> {{left, right}, {bottom, top}}]];**

(*This function scans to length 12 on each edge with variable depth. The grid below shows the webs for the regular pentagon (N = 5) as the depth varies from 0 to 7.*) **box[{0,0},9];**

**GraphicsGrid[{{Gr[0],Gr[1],Gr[2],Gr[3]},{Gr[4],Gr[5],Gr[6],Gr[7]}},Frame->All]**

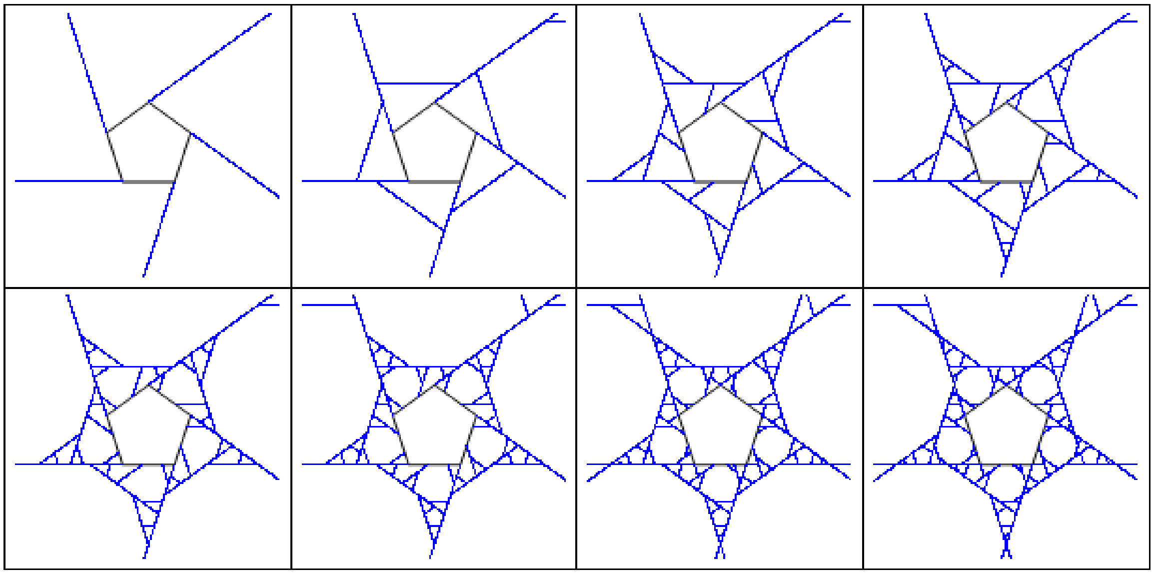


Below are the identical web plots for the N5Kite. The ring of 5 'Dads' has gaps so the central 'star' region is no longer invariant. The second ring of Dads will be even more irregular.

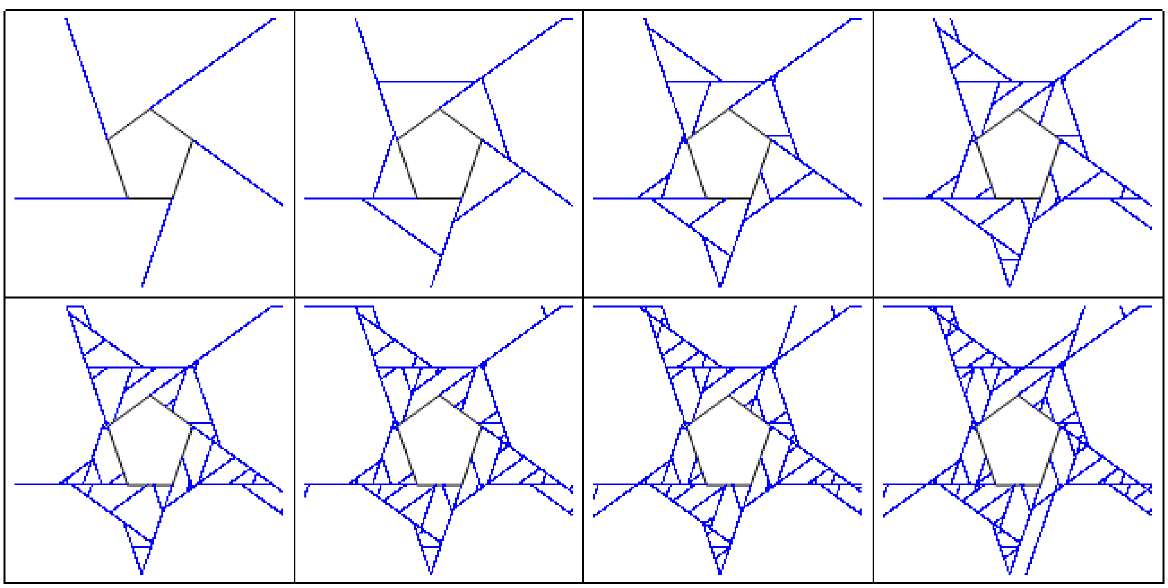


Below are the same webs at a closer scale: **box[{0,0}, 3];**

Regular pentagon N = 5, web scans of depth 0 to depth 7



The N5Kite webs below show irregular 'star' formation which prevents it from ever becoming invariant. The 'Dads' edges never seal off the boundary and instead 'channels' form which provide conduits for points to diverge. Many points close to Mom diverge.

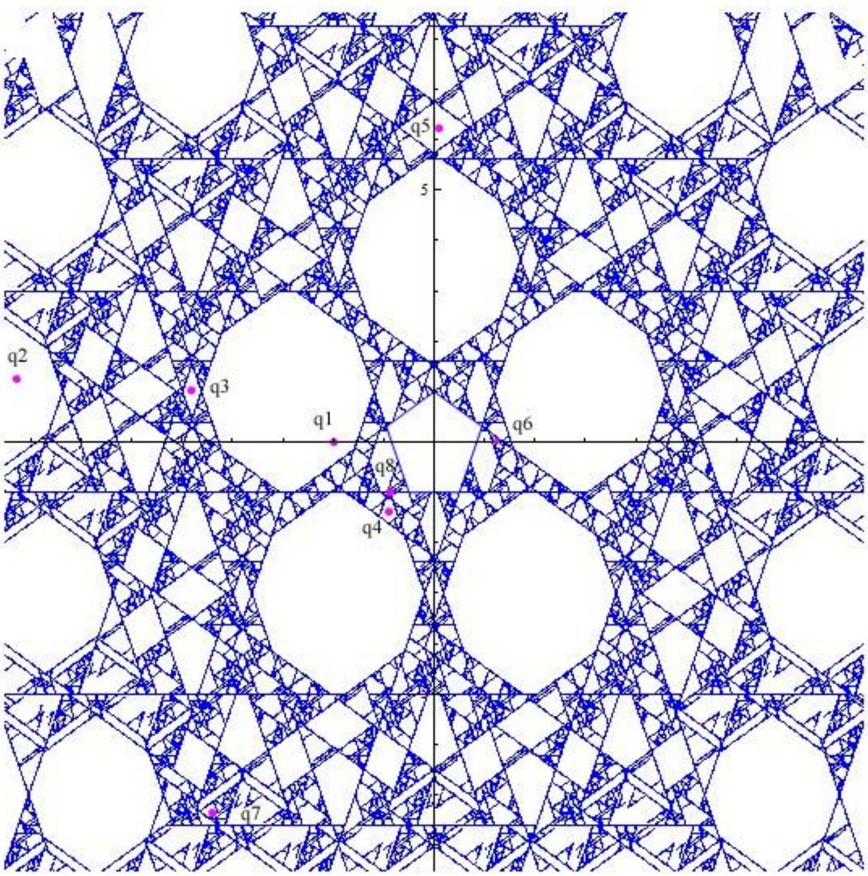


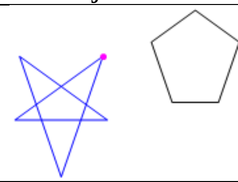
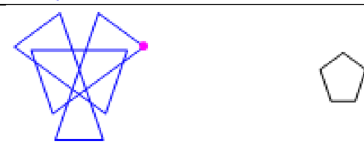
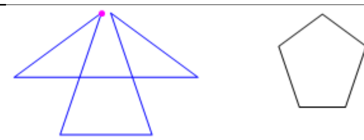
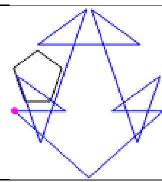
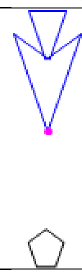
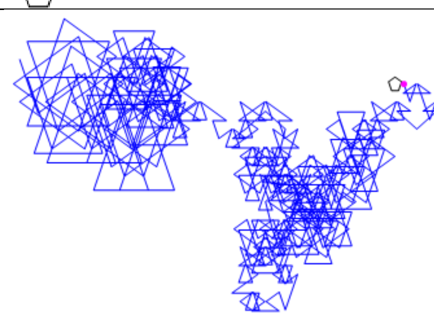
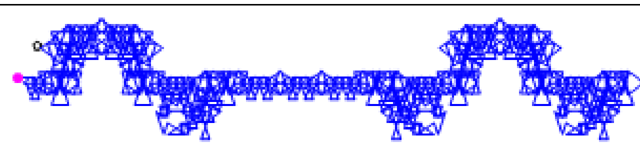
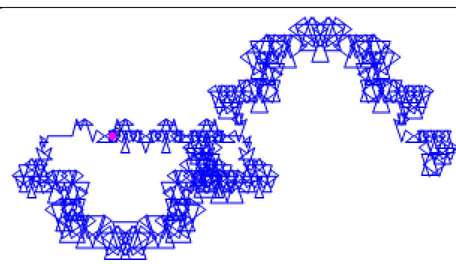
For any polygon, the web will continue to evolve at larger distances from the origin, but frequently the global evolution can be understood based on the local evolution. For lattice polygons, the web will stabilize in any bounded region, indicating that all orbits are periodic.

For most regular polygons, the web will always evolve, even within the invariant 'star' region. But for the regular pentagon, the evolution is self-similar which makes it easy to predict the limiting web structure and to find limit points with non-periodic orbits. For the remaining regular polygons, the central 'star' region is still invariant and it provides a 'template' for the global evolution, so the global evolution can be predicted from the local.

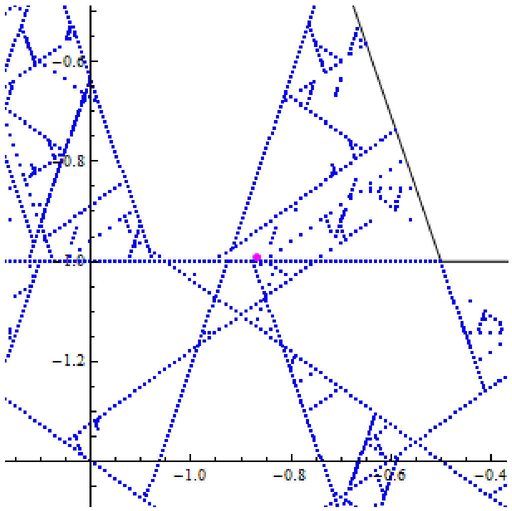
For the N5Kite it is not possible to distinguish the global evolution from the local evolution so we have no clue about the limiting structure. It is almost certainly the case that there are regions which will remain unresolved no matter what depth is used for the scan - because they contain points with unbounded orbits. Just as in the regular case, there are periodic regions that will not evolve further, but there are also many regions which defy further resolution.

Below is a depth 1000 web for the N5kite. The table below shows the coordinates, periods and projections for the 8 marked points. The points q6 and q8 may have unbounded orbits, and we do not know the period of p7.



| Point | Period | P2 Projection - showing Mom for perspective |
|---|---|---|
| q1 = {-2,0} | 10 (center has period 5) |  |
| q2 = {-8.287, 1.254} | 30 (center has period 15) |  |
| q3 = {-4.803,1.026} | 14 |  |
| q4 = {-0.8954, -1.384} | 32 |  |
| q5 = {0.0814, 6.236} | 18 |  |
| q6 = {1.275,-.001}. | Unknown (first 1500 P2 points shown here) |  |
| q7 = {-4.412, -7.343} | Unknown (first 1500 P2 points shown here) |  |
| q8 = {-.87,-.99}; | Unknown (first 1500 P2 points shown here.) |  |

One prime candidate for an unbounded orbit is q8 = {-.87,-.99} from above. It was chosen because it comes from a region which is almost self-similar to the star region. That region is shown below with q8 in magenta and Mom on the right.




Divergent orbits such as q8 or q6 = {1.275,-.001} can be used to see how the web structure evolves away from the origin. To do this, choose a primary 'strip' and crop the orbit to this strip. As orbits diverge the dynamics in any strip tend to be similar. There is a natural width for the primary strip for the N5Kite - which is connected to Pinwheel dynamics:

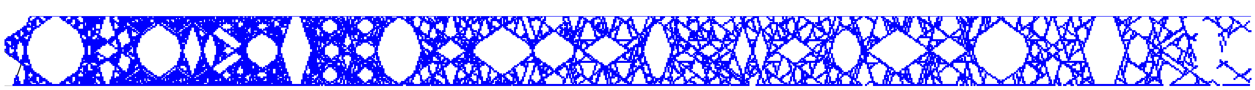
top = 3; bottom = -1;

For example: **left = 0; right = 200; top = 3; bottom = -1;**

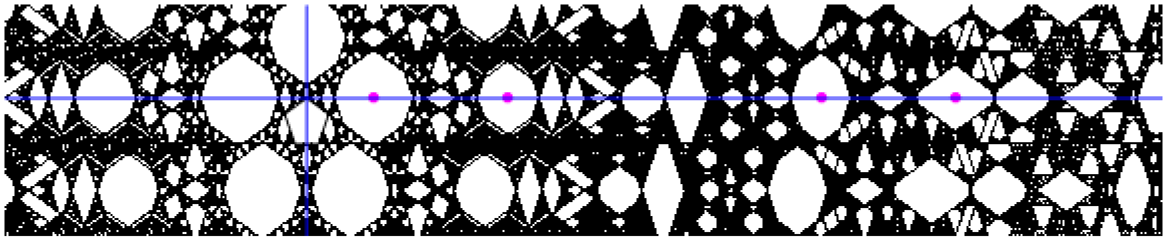
**Datumcrop[q6, 50, 1000000,"temp"]** (*This will do 50 iterations of 1 million each and crop the resulting points after each iteration. It will print results after each iteration and the process can be aborted at any time without loss of data.*)

This command generated 34111 points as shown below:

Below is an enlargement of the beginning of this strip.



To track the evolution of the web,we have used billions of iterations to create detailed strips such as the one shown below. It is widened to show more of the ring evolution. The data for these strips was gathered from 4.5 billion iterations over a period of 3 weeks. We also tracked distances and winding numbers for q8 and q6.

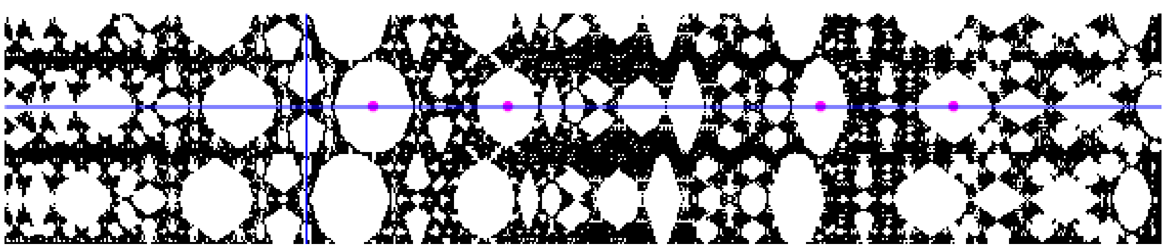


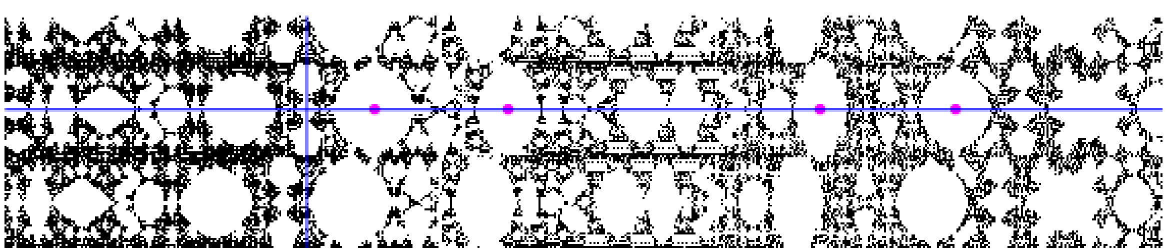
The horizontal blue line at y = 1 is a major center line for the Dads. The red dots mark the first four rings of 'Dads'. The (exact) centers are: c1 = {2.9,1}, c2= {8.7,1}, c3 = {22.2,1}, c4= {28,1}. These can be found with high precision because the Dads have period doubling.

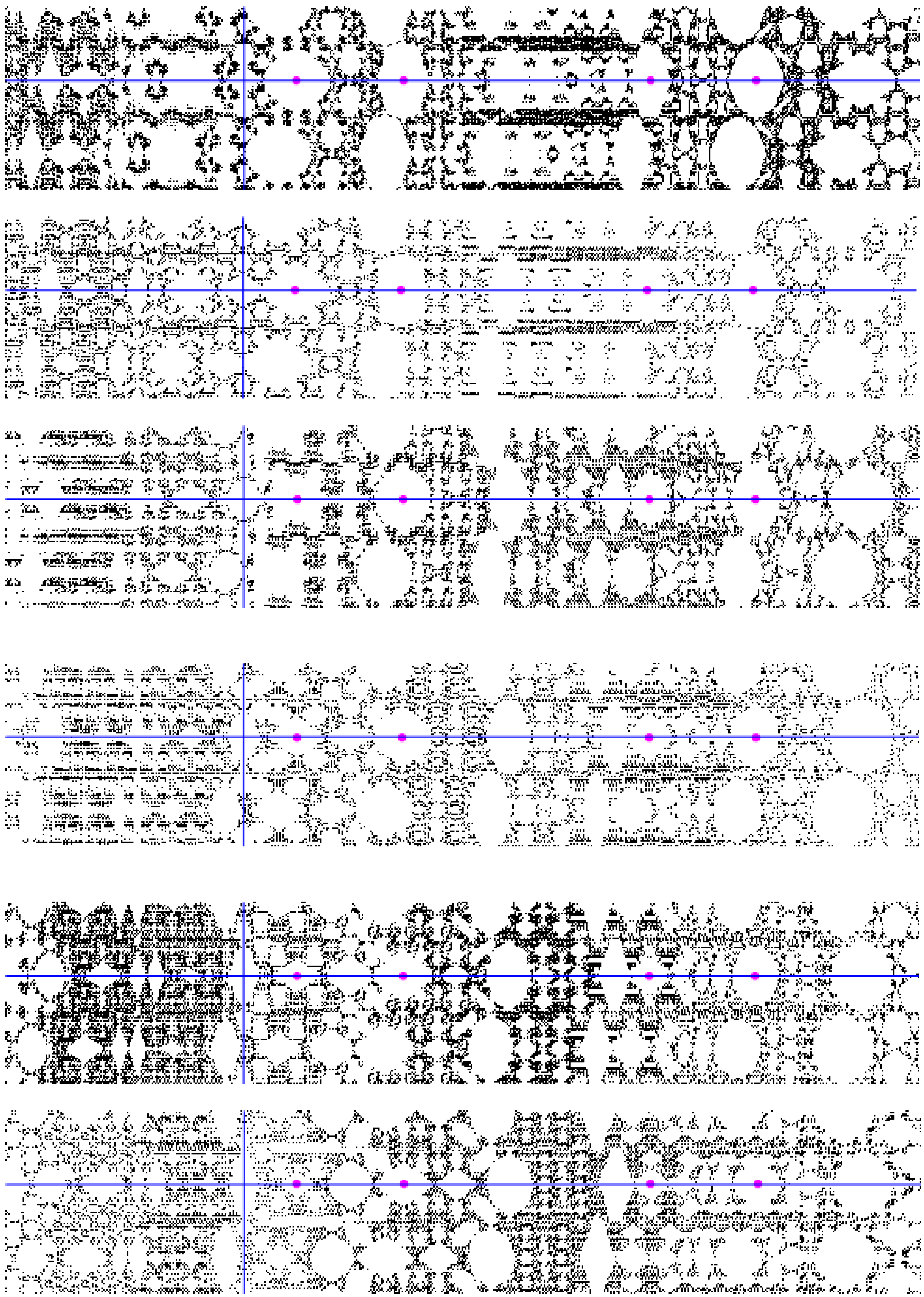
There are actually two groups of two rings each, with interval spacing 5.8 in each case. The spacing shown here is somewhat 'canonical' for subsequent rings. If Mom was regular, the '5.8' spacing would continue for all rings, and the periods of the centers would be 5 + 10k for ring k. What we find here is an irregular ring structure but there appears to be a major cycle length of 299.3 where the periods of the c1 centers are 5 + 498k, so each of these major cycles acts like a 'super-ring'.

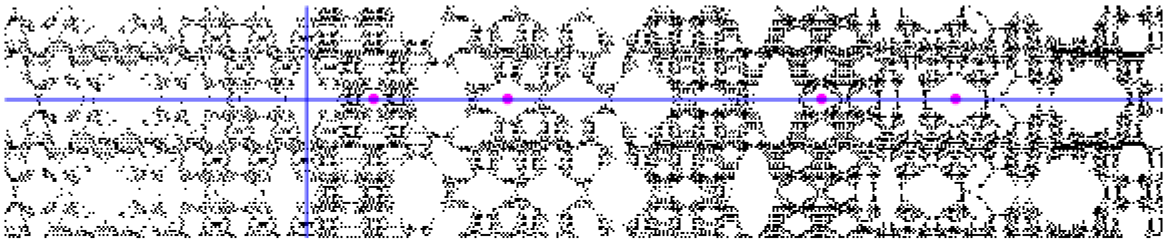
For example the seventh c1 surrogate with coordinates {2.9 + 7* 299.3,1} has period 5 + 7*498. The same applies to c2, c3 and c4 and other ring centers. The c2 surrogate with coordinates {8.7 + 7*299.3,1} has period 15 + 7*498 since c2 has period 15. If we track the origin in this fashion it is not as 'robust'. The periods are 996k for k ≤ 4 but they jump to 2 + 996k and maintain this ratio up to 20 where they make a large jump. The c1center requires a minor adjustmen of 2 at cycle 12 and at cycle 25 it breaks down. c2 is somewhat more 'stable' but all points tracked undergo some shifts - however the asymptotic ratios appear to approach 996. There may be some points in the initial cycle from 0 to 299.3, where periods always increase by 996. The images below show how the region evolves as centers disappear and others take their place.

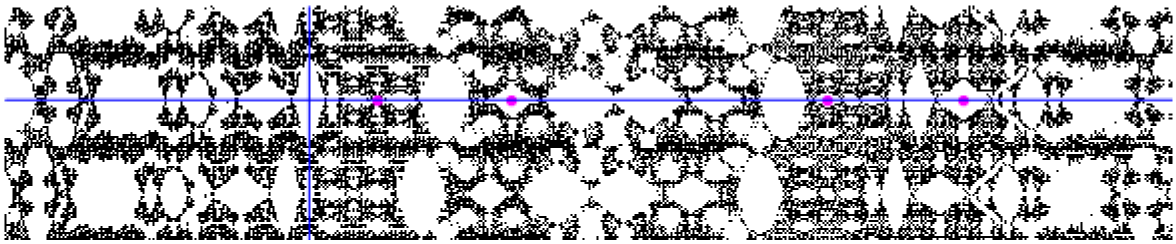
The 11 snapshots shown below are taken at intervals of 299.3 and we can see the gradual deterioration of the centers c1,c2,c3 and c4. The length of each strip is 50 units, so it just shows a portion of each 'super-ring'. While these initial rings are being destroyed, new rings are forming and we can see this clearly in the last snapshot at cycle12.

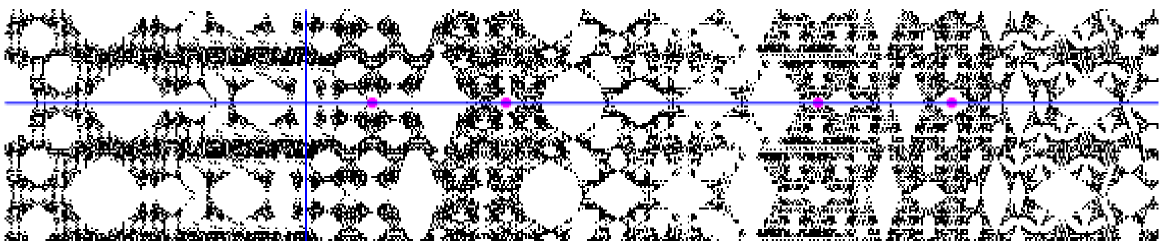






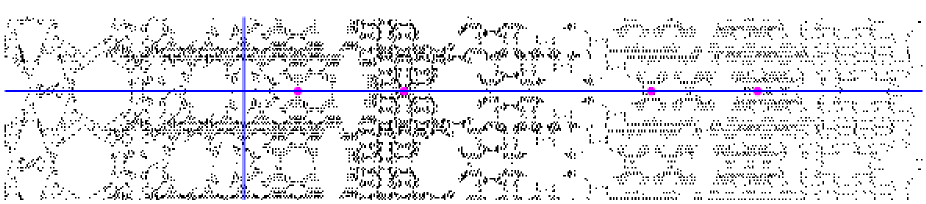




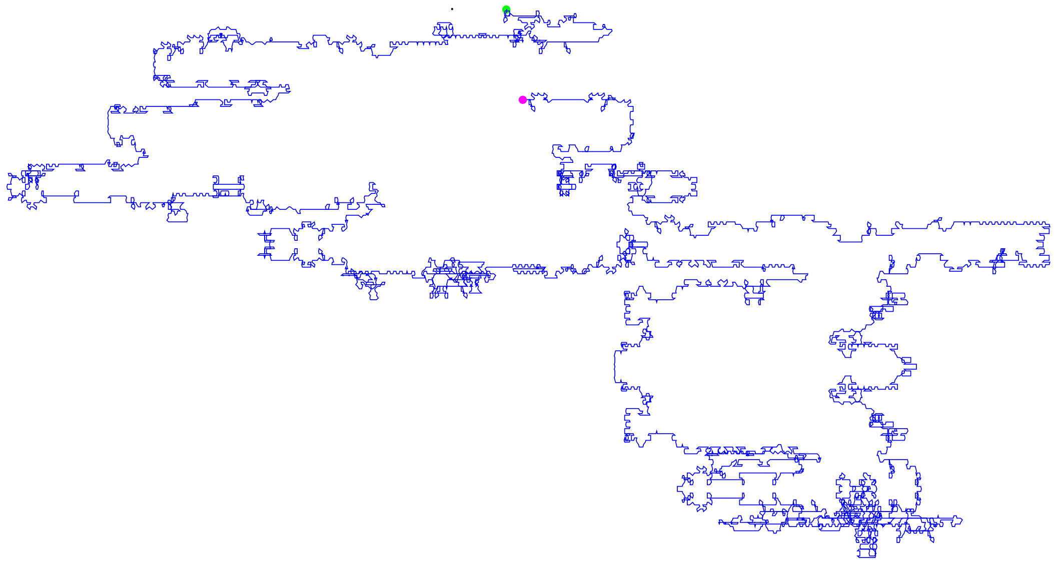


The last cycle shown below is 12 and on the left we can see the emergence of a new 'Mom-Dad' structure centered at x = 3581 (exactly). This is actually the latter portion of the pervious 11th cycle.

Meanwhile the surrogate center c1 = {2.9+ 299.3*12,1} has almost disappeared and the period shows an increase of 500 instead of the expected 498. The subsequent 12 cycles show a return to the 498 interval, but on the 25 cycle there is a major increase in the period. This may indicate that the deterioration of this 'c1' region does not reverse itself. Since we expect that the overall ring structure is non-periodic, there seems little chance of a periodic recurrence of the initial centers.We have not explored the detailed dynamics past the 12th 'super-ring' because the current probes are at a maximum displacement of 3900 after 4.5 billion iterations.



Below is a typical P2 Pinwheel projections for the orbit of q6 = {1.275,-.001}. This is the orbit that generated most of the plots above and we have tracked it out to a distance of about 4000 units. This plot starts at {1209.13, 0.741861} (in the strip) and runs for 7655 iterations. In the Pinwheel map the P1 points are confined to the strip but the P2 points can wonder freely. (Green is start, magenta is stop.)



Just like the orbits, these projections have the habit of doubling back as if to erase their tracks. Below is the continuation of this P2 projection. The magenta stop sign is just visible in the center top.