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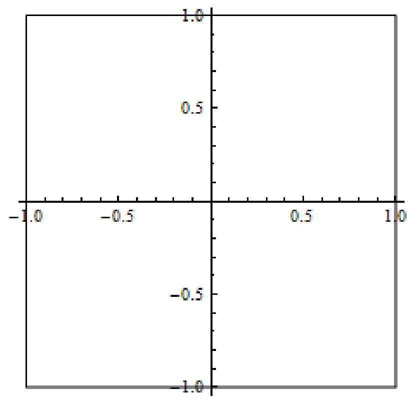
Summary of dynamics of the square: N = 4

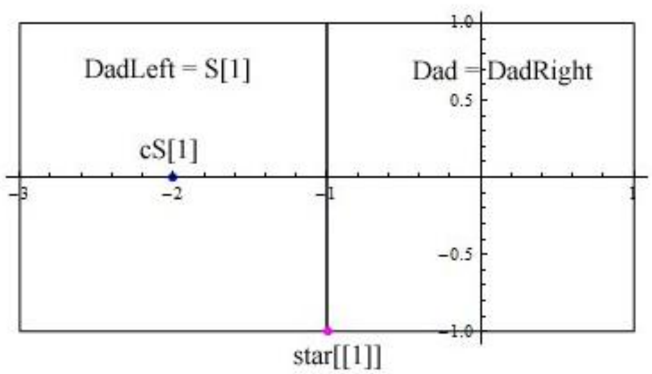
The square is the second regular lattice polygon - along with the equilateral triangle and the regular hexagon. Since our convention for 2k-gons is a height of 1, Dad for N = 4 will have vertices {1,1},{1,-1},{-1,-1}, {-1,1}. The notebook NTwiceEven makes no distinction between Mom and Dad, but we will usually call 2k-gons Dads.

Graphics[poly[Dad], Axes->True]



The First Family for N = 4 is shown below. It looks a little strange because we usually include DadLeft and DadRight (Dad) in the family. In the case of N = 4, these are the only 2 family members.

Graphics[{poly/@FirstFamily, Magenta,AbsolutePointSize[5.0],Point[star[[1]]], Blue,Point[cS[1]]},Axes->True]

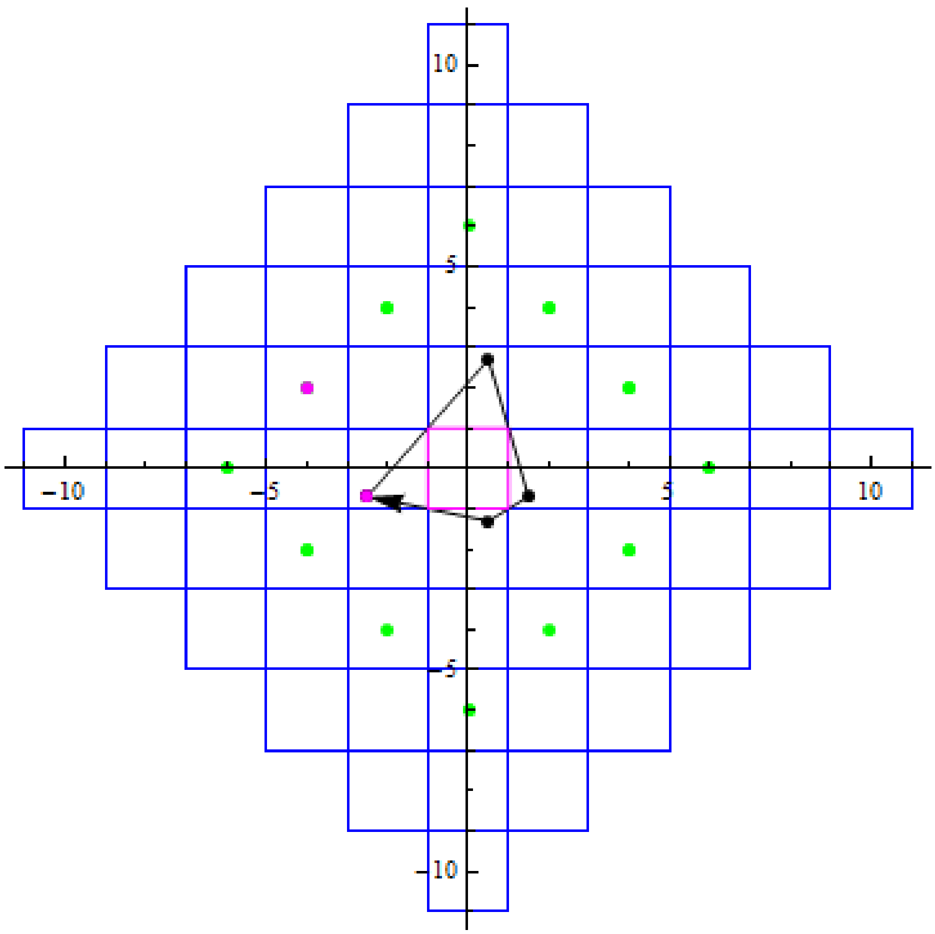


The magenta point is star[[1]], the only star point. The star points define centers, which in this case is cS[1] shown in blue. The centers then define family members, so DadLeft is known as S[1] because he is step-1 relative to Dad. For any point in the interior of S[1], the orbit around Dad skips one corner on each iteration before returning home after 4 iterations. So the orbit is period 4 and there is a ring of 4 identical S[1]'s surrounding Dad. One such orbit is shown below overlaid on a web plot.

web[.01,10,100,0];WebPlot={};For[i=1,i<=npoints,i++,WebPlot = Union[WebPlot,Jxy[[i]]]];

q1= {-2.5,-.7}; K = V[q1,4]; (*First 4 points in the orbit of q1 It repeats after 4 points*)

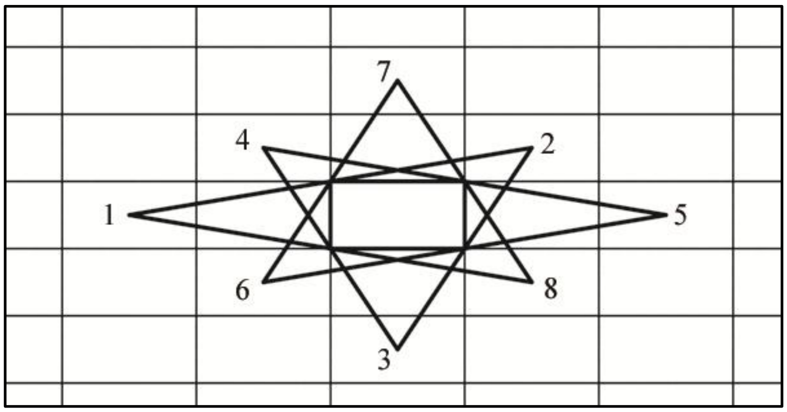
Graphics[{AbsolutePointSize[1.0], Arrow[K], AbsolutePointSize[5.0],Point[K] Blue,Point[WebPlot],Magenta,poly[Dad], AbsolutePointSize[5.0], Point[q1]},Axes->True]



The web is in blue and the initial point q1 is magenta. The second magenta dot is in ring 3 and we can see that it has period 12. The coordinates of this point are {-4,2} and since this is a lattice polygon every point in the orbit will have integer coordinates, for example τ({-4,2}) = {6,0}. This makes it easy to calculate orbits by hand with no round-off.

Ring k will have 4k Dads and this is also the period. In terms of step sequences, the first ring will have step sequence (1) and the points in the second ring will have step sequence (12) and each additional ring will add another 2 to the sequence, so the limiting sequence will be (2).

Any rectangle or affine transformation of a square, would have the same dynamics and periods. Shown below is a rectangle with width 2 and side 1/2. The orbit shown is the second ring of Dads so it is period 8.



For quadrilaterals which are not affinely equivalent to a square, such as the Penrose kite, there may be unbounded orbits, but D.Genin has shown that a trapezoid always has bounded orbits

2008: . D. Genin, *Research announcement: boundedness of orbits for trapezoidal outer billiards.* Electronic Research Announc. Math. Sci. **15** , 71–78. MR2457051 (2009k:37036)

The web for an arbitrary trapezoid is shown below. There is no affine transformation that would turn such a trapezoid into a kite.