Families of Regular Polygons and their Mutations

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Abstract

Every regular N-gon generates a canonical ‘family’ of regular polygons which are conforming to the bounds of the ‘star polygons’ determined by N. These star polygons are formed from truncated extended edges of the N-gon and the intersection points (‘star points’) determine a scaling which defines the parameters of the family. Based on a 1949 result of C.L. Siegel communicated to S. Chowla [Ch], it follows that this star-point scaling forms a unit basis for $\mathbb{Q}_N^+$ the maximal real subfield of the cyclotomic field $\mathbb{Q}_N$. The traditional generator for this subfield is $\lambda_N = 2\cos(2\pi/N)$ so it has order $\varphi(N)/2$ where φ is the Euler totient function. This order is known as the ‘algebraic complexity’ of N. The family of conforming regular polygons shares the same scaling and complexity as N, so it is called the First Family of N.

Each star[k] point of N defines a scale[k] and also an S[k] ‘tile’ of the First Family. Because these S[k] are regular polygons, their First Families can be used to define a recursive nesting with known scaling. This scaling would typically lead to a multi-fractal topology for the star polygons of N, and the scaling parameters would be units in the ‘scaling field’ $\mathbb{Q}_N^+ = \mathbb{Q}(\lambda_N)$, so calculations would be efficient and exact. (Some of these calculations will be carried out here.)

This scenario exists under a piecewise isometry such as the outer-billiards map τ because the ‘singularity set’ W can be formed by iterating the extended edges of N under τ and we show in Section 4 that W preserves the S[k]. Therefore this ‘web’ W can be regarded as the disjoint union (coproduct) of the local webs of the S[k]. These local webs are still very complex but for N-gons with complexity that is linear or quadratic (N = 3,4,5,6,8,10,12), this S[k] evolution is sufficient to describe the resulting topology of W - because in these cases there is never more than one effective scale. Therefore the topology is linear for N = 3, 4 and 6 and simple fractal for N = 5,8,10 and 12 - which are the only S[k] with quadratic complexity.

When there are multiple non-trivial scaling parameters, the topology of W may be multi-fractal and there is no guarantee that the First Family scaling will be sufficient to determine the geometric scaling of W - but we conjecture that this scaling in $\mathbb{Q}_N^+$ is always sufficient.

Organization of the five sections of this paper (see the Table of Contents)

- Section 1 introduces star polygons and their scaling. The main results are the Scaling Lemma for nested regular polygons and the Two-Star Lemma for construction of regular polygons.

- Section 2 contains a geometric and algebraic derivation of the S[k] tiles which make up the First Families. The main results are the First Family Theorem and the First Family Scaling Lemma which says that hS[1]/hS[k] = scale[k] – so S[1] can be used as a reference for scaling.

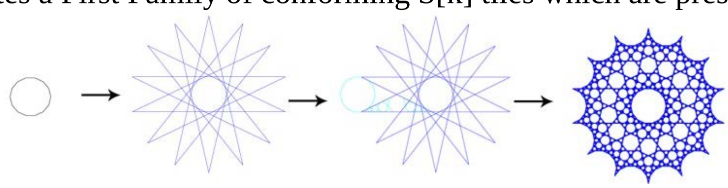
- Section 3 is devoted to GenScale[N] – which is the scaling of S[1] relative to the maximal S[k] tile - called D. We show that this scaling is actually inherent in the star polygons of N, so every regular N-gon has the potential to support ‘generations’ of First Families on the edges of D. The Scaling Field Lemma shows that GenScale[N] (or GenScale[N/2]) are unit generators of the scaling field $\mathbb{Q}_N^+$.This provides a natural scaling for tiles in the complement of W.

• Section 4 relates this geometry and scaling to the outer-billiards map τ. Lemma 4.1 makes use of a height/radius duality between the star[k] points and S[k] centers to show that any edge-based star polygon orbit generates a vertex-based τ orbit. Since these star polygon orbits are mod-k, each S[k] defines a periodic step-k τ orbit which mimics the 'resonances' of Hamiltonian dynamics. This connection between geometry and dynamics is difficult to extend to the remaining tiles of W, but the constant step-k orbits of the S[k] set bounds on the possible step-sequences. This 'symbolic dynamics' of orbits will be discussed in the Appendix.

Example 4.3 shows how the evolution of the web W can be reduced to a simple 'shear and rotate'. This implies that the S[k] evolve in a multi-step fashion and explains possible mutations. We also show how to decompose W into a disjoint union of local S[k] webs. These local webs are still very complex but the Edge Conjecture gives explicit predictions about the web evolution of the important S[1] and S[2] tiles which are adjacent to N. This provides a plausible explanation of the long-standing '4k+1' conjecture of [H3] about the existence of generations of S[1] and S[2] tiles when N is of the form 8k+2. Each 8k+j family has distinct edge geometry.

• Section 5 has annotated examples of singularity sets and the Appendix describes the 'digital-filter' map of Chau & Lin and a 'dual-center' map of Erik Goetz. Both of these maps reduce to a 'shear and rotation' and their webs appear to replicate W. We will use symbolic dynamics relative to τ and these two maps to attack the same problem involving the web evolution of a tile called Mx which is 'weakly conforming' to N=11.

Below is a graphical summary of this paper, showing how the regular tetradecagon known as N =14 generates a First Family of conforming S[k] tiles which are preserved by W.



Historical Background

In a 1978 article called "*Is the Solar System Stable*?" Jurgen Moser [Mo2] used the landmark KAM Theorem – named after A. Kalmogorov,V. Arnold and Moser – to show that there is typically a non-zero measure of initial conditions that would lead to a stable solar system – but these initial conditions are unknown and this is still an open question. Since the KAM Theorem was very sensitive to continuity, Moser suggested a 'toy model' based on orbits around a polygon – where continuity would fail. This is called the 'outer-billiards' map and in 2007 Richard Schwartz [S1] showed that if the polygon was in a certain class of 'kites', orbits could diverge and stability failed. The special case of a regular polygon was settled earlier in 1989 when F.Vivaldi and A. Shaidanko [VS] showed that all obits are bounded.

The author met with Moser at Stanford University in that same year to discuss the 'canonical' structures that always arise in the regular case - and Moser suggested that a study of these structures would be an interesting exercise in 'recreational' mathematics. This is exactly what it became over the years - but the evolution of these canonical 'First Families' proved to be a difficult issue except for regular N-gons with linear or quadratic complexity.

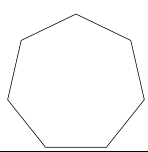
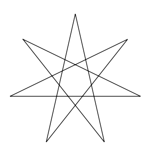
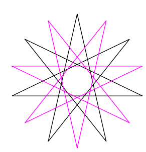
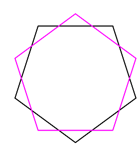
Section 1. Star Polygons and Star Points

'Star' polygons or 'stellated' polygons were first studied by Thomas Bradwardine (1290-1349), and later by Johannes Kepler (1571-1630).

The vertices of a regular N-gon with radius r are {rCos[2πk/N], rSin[2πk/N]} for {k,1,N}. A 'star polygon'{p,q} generalizes this by allowing N above to be rational of the form p/q so the vertices are given by {p,q} = {rCos[2πkq/p], rSin[2πkq/p]} for {k,1,p}

Using the notation of H.S.Coxeter [Co] a regular heptagon can be written as {7,1} (or just {7}) and {7,3} is a 'step-3' heptagon formed by joining every third vertex of {7} so the exterior angles are 2π/(7/3) instead of 2π/7.

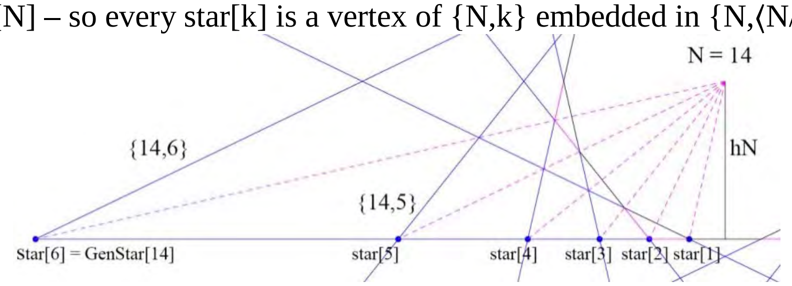
By the definition above,{14,6} would be the same as {7,3}, but there are two heptagons embedded in N = 14 and a different starting vertex would yield another copy of {7,3} - so a common convention is to define {14,6} using both copies of {7,3} as shown below. This convention guarantees that all the star polygons for {N} will have N vertices.

{7,1} (a.k.a. N = 7)	{7,3}	{14,6}	{10,2}
			

The number of 'distinct' star polygons for {N} is the number of integers less than N/2 – which we write as ⟨N/2⟩. So for a regular N-gon, the 'maximal' star polygon is {N, ⟨N/2⟩}.

Our default convention for the 'parent' N-gon will be centered at the origin with 'base' edge horizontal, and the matching {N,1} will be assumed equal to N. In general sN, rN and hN will denote the side, radius and height (apothem). Typically we will use hN as the lone parameter.

Definition 1.1 The *star points* of a regular N-gon are the intersections of the edges of {N, ⟨N/2⟩} with a single extended edge of the N-gon (which will be assumed horizontal). By convention the star points are numbered from star[1] (a vertex of N) outwards to star[⟨N/2⟩] – which is called GenStar[N] – so every star[k] is a vertex of {N,k} embedded in {N,⟨N/2⟩}.



The star points could equally be defined on the positive side of N, but this 'left-side' convention is sometimes convenient. The symmetry of these choices makes it irrelevant which convention is used. In general, the star points of a regular N-gon with apothem hN are :

star[k] = ±hN·(s_k,1) where s_k = Tan[kπ/N] for 1 ≤ k < N/2.

The *primitive* star[k] and s_k are those with (k,N) = 1, where (k,N) is gcd(k,N). Sometimes we will loosely refer to the s_k as 'star' points.

Note: There is a long history of interest in trigonometric functions of rational multiples of π. Niven [N] shows that these 'trigonometric numbers' are also algebraic numbers. In 1949, C.L. Siegel communicated to S.Chowla [Ch] a proof that the primitive s_k for the cotangent function are linearly independent. This was a non-trivial result in algebraic number theory and Siegel only proved it for prime N. In 1970 Chowla generalized this result using character theory and Dirichlet's $\mathcal{L}$-series, but the general tangent case was only settled recently in [Gi] (1997). Therefore the primitive star points are linearly independent and it follows that any non-primitive s_k must be a $\mathbb{Q}$-linear combination of the primitive s_k. See Corollory 3.2 to Lemma 3.2

Section 3 is devoted to scaling, but here we present the basic definitions and prove the Scaling Lemma. In Section 3 we will use these scales as a basis for the maximal real subfield of $\mathbb{Q}_N$.

Definition 1.2 The (*canonical) scales* of a regular N-gon are scale[k] = s_1/s_k for $1 \leq k < N/2$ .The *co-scales* are of the form s_k/s_1. The *primitive* scales or coscales are those with (k,N) = 1. GenScale[N] is scale[<N/2>].

By definition scale[1] is always 1 and GenScale is the minimal scale. Since these scales are independent of height, to compare scales for an N-gon and an N/k-gon, the later can be regarded as circumscribed about the N-gon with shared center and height – so the sides can be compared.

Definition 1.3 If N and M are regular polygons with M = N/k, then ScaleChange(N,M) = (s_1 of N)/(s_1 of M) ≤ 1. This is abbreviated SC(N,M) or just SC(N) < ½ ,when M is N/2.

Lemma 1.1 (Scaling Lemma) Suppose N and M are regular polygons and M = N/k , then

scale[j] of N/k = scale[kj] /scale[k] of N

Proof: There is no loss of generality in assuming that N and N/k are in 'standard position' at the origin with equal heights so N/k will be a 'circumscribed factor polygon' of N. If s is the side of N, then s/SC(N,N/k) will be the side of N/k. The external angle of N/k is $2\pi k/N$ so in this position, every kth edge will coincide with an edge of N. Therefore it will share every kth star point with N and by definition the corresponding scales are related by the ratio of the sides of N and N/k, so scale[j] of N/k = (scale[kj] of N)/SC(N,N/k). Since scale[1] of N/k = 1 = scale[k]/SC(N,N/k), it follows that scale[k] of N is SC(N,N/k). □

Therefore scale[k] of N = 7 is the same as scale[2k]/scale[2] for N = 14 and this twice-odd case is the only nesting where the GenStar points coincide, so GenScale[7] = GenScale[14]/scale[2].

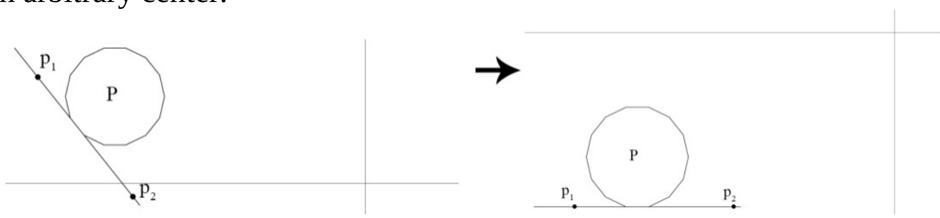
Lemma 1.2 When N is even, GenScale[N] = $\mathrm{Tan}^2[\pi/N] = s_1^2$ and when N is odd GenScale[N] = $\mathrm{Tan}[\pi/2N]\cdot\mathrm{Tan}[\pi/N] = s_1 s_2$ of 2N.

Proof : For N even, the star points inherit the reflective symmetry of N under $\tan[\pi/2 - \theta] = \cot[\theta]$, so $s_{N/2-k} = 1/s_k$. Setting k = <N/2> = N/2-1, GenScale[N] = $s_1/(1/s_1) = s_1^2$. When N is odd the Scaling Lemma says that GenScale[N] = GenScale[2N]/scale[2] = $s_1^2/(s_1/s_2) = s_1 s_2$ of 2N. □

To define a regular N-gon in a given coordinate system, it is sufficient to know its height (apothem) and its center, but both of these are determined by knowing the co-ordinates of two star points – as any cartographer would know.

Lemma 1.3 (Two-Star Lemma) If P is a regular N-gon, any two star points are sufficient to determine the center and height.

Proof. By definition, the star points lie on an extended edge of P. There is no loss of generality in assuming that this extended edge is parallel to the horizontal axis of a known coordinate system with arbitrary center.



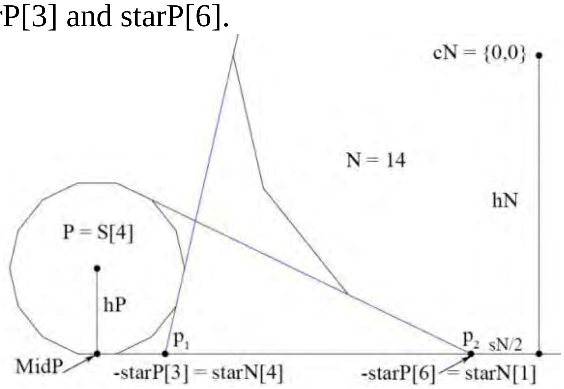
Since all points on this extended edge will have a known second coordinate, we will just need the horizontal coordinates of the star points – which we will call p_1 and p_2 with $p_2 > p_1$, so $d = p_2 - p_1$ will be positive. Relative to P, $p_1 = \pm$ star[j][[1]] = $\pm$hP·Tan[jπ/N] and $p_2 = \pm$ star[k][[1]] = $\pm$hP·Tan[kπ/N] (where p[[1]] is the horizontal coordinate of p).These indices j and k must be known. There are only two cases to consider:

(i) If p_1 and p_2 are on the same side of P, there is no loss of generality in assuming that it is the right-side of P because star points always exist in their symmetric form with respect to P. In this case we can assume that $1 \leq j < k < N/2$ so hP = d/(Tan[kπ/N]-Tan[jπ/N]).

(ii) If p_1 and p_2 are on opposite sides hP = d/ (Tan[kπ/N] + Tan[jπ/N]) and it does not matter whether $j \leq k$ or not.

Now that hP is known, the horizontal displacement of p_1 and p_2 relative to P are :
x = hP·Tan[jπ/N] and (x +d) = hP·Tan[kπ/N] if both are on the same side
or x = hP·Tan[jπ/N] and (d-x) = hP·Tan[kπ/N] if they are on opposite sides
Of course only one of these displacements is needed to define the center of P. □

Example 1.1 (The one-elephant case) P below shares two star points with the elephant N = 14 – which defines the coordinate system. The shared star points are star[3] and star[6] of P which we abbreviate as starP[3] and starP[6].



Because of the symmetry between P and N, either one could be used as reference to construct the other – but here we assume that hN is known and it is desired to find hP (and cP) relative to N.

By the Two-Star Lemma hP = $\dfrac{p_2 - p_1}{Tan[6\pi/14] - Tan[3\pi/14]} = hN\dfrac{Tan[\frac{4\pi}{14}] - Tan[\frac{\pi}{14}]}{Tan[\frac{6\pi}{14}] - Tan[\frac{3\pi}{14}]}$

Therefore $\dfrac{hP}{hN} = \dfrac{hS[4]}{hN} = \dfrac{Tan[\frac{4\pi}{14}] - Tan[\frac{\pi}{14}]}{Tan[\frac{6\pi}{14}] - Tan[\frac{3\pi}{14}]} = \dfrac{Tan[\pi/14]}{Tan[3\pi/14]} = \dfrac{s_1}{s_3}$ (which is also s_1s_4)

Therefore hP $\cdot s_3$ = hN$\cdot s_1$. By definition, the right side is the horizontal coordinate of –star[1] of N which is –sN/2. This must be equal to the left side which is the horizontal coordinate of +star[3] of P. This is measured relative to MidP, so the displacement of MidP from star[3] must be –sN/2. This also follows from the symmetry between P and N. Since P can construct N (by setting $p_1 = -p_2$), the two displacements must be equal.

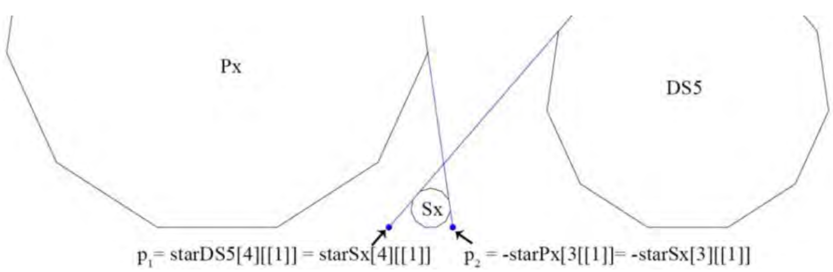
P here is known as S[4] because it was constructed using star[4] (and star[1]) of N. Clearly this same construction can be carried out for the remaining star[k] points of N = 14 – and we will do this below for arbitrary N. Since each S[k] will have a symmetric relationship with N, the displacements will always be –sN/2. (And each S[k] will have a natural embedding in N.)

Algebraically the S[k] and N are closely related because hP/hN will always be an element of the 'scaling field' defined by N = 14 (or N = 7). This is the number field generated by 2Cos[2π/7] or GenScale[7] = Tan[π/7]·Tan[π/14] . See Section 3. Using Mathematica:

AlgebraicNumberPolynomial[ToNumberField[hS[4]]/hN,GenScale[7]],x] = $\dfrac{x^2 - x + 1}{2}$

So hS[4]/hN = $\dfrac{x^2 - x + 1}{2}$ where x = Tan[π/7]·Tan[π/14] and this scale is an algebraic number.
Since N= 14 and the matching N = 7 have φ(N)/2 = 3 where φ is the Euler totient function, they are classified as 'cubic' polygons. This means that any generator of the scaling field will have minimal degree 3, so the scaling polynomials will be at most quadratic.

Example 1.2 (The two-elephant case) The tiles below exist in the coordinate space of N = 11. This is a 'quintic' N-gon so the algebra is much more complex than N = 14. Px and DS5 share a star point which is off the page at the right, but they do not share any other star points so it was a challenge to find a second defining star point of Px – even though the parameters and star points of DS5 are known from the First Family Theorem to follow. Since these two elephants are only distantly related, it is unusual for them to share a third tile – which we call Sx. This Sx tile shares extended edges with both Px and DS5.



These tiles share the 'base' edge of N = 11. The calculations below assume hN = 1 so the vertical coordinate of the star points is -1 and we will just need the horizontal coordinates .

$$p_1 = starDS5[4][[1]] = -\tan\left[\frac{\pi}{22}\right]^2\left(2\cot\left[\frac{\pi}{22}\right]-\cot\left[\frac{\pi}{22}\right]^3+\tan\left[\frac{\pi}{11}\right]\right)+2\cot\left[\frac{\pi}{22}\right]\left(-1+2\tan\left[\frac{\pi}{22}\right]^2+\tan\left[\frac{\pi}{22}\right]^3\tan\left[\frac{\pi}{11}\right]\right)-$$

$$\cot\left[\frac{3\pi}{22}\right]\tan\left[\frac{\pi}{22}\right]^2\tan\left[\frac{\pi}{11}\right]\left(2+\tan\left[\frac{\pi}{22}\right]\tan\left[\frac{\pi}{11}\right]\right)\tan\left[\frac{5\pi}{22}\right]+\tan\left[\frac{\pi}{11}\right]\left(1-\tan\left[\frac{\pi}{22}\right]\left(2+\tan\left[\frac{\pi}{22}\right]\tan\left[\frac{\pi}{11}\right]\right)\tan\left[\frac{5\pi}{22}\right]\right)$$

The trigonometric expression for star[3] of Px is more complex because Px shares no canonical scaling with N. Mathematica prefers to do these calculations in 'cyclotomic' form – which will have vanishing complex part here. The cyclotomic form for star[3] of Px is:

$$p_2 = -starPx[3][[1]] = -\frac{i\left(3-8(-1)^{1/11}+6(-1)^{2/11}+4(-1)^{3/11}+11(-1)^{4/11}+11(-1)^{5/11}+4(-1)^{6/11}+6(-1)^{7/11}-8(-1)^{8/11}+3(-1)^{9/11}\right)}{\left(1+(-1)^{2/11}\right)^2\left(-1+(-1)^{5/11}\right)}$$

By the Two-Star Lemma, hSx = (p_2-p_1)/(Tan[4π/11]+Tan[3π/11]) =

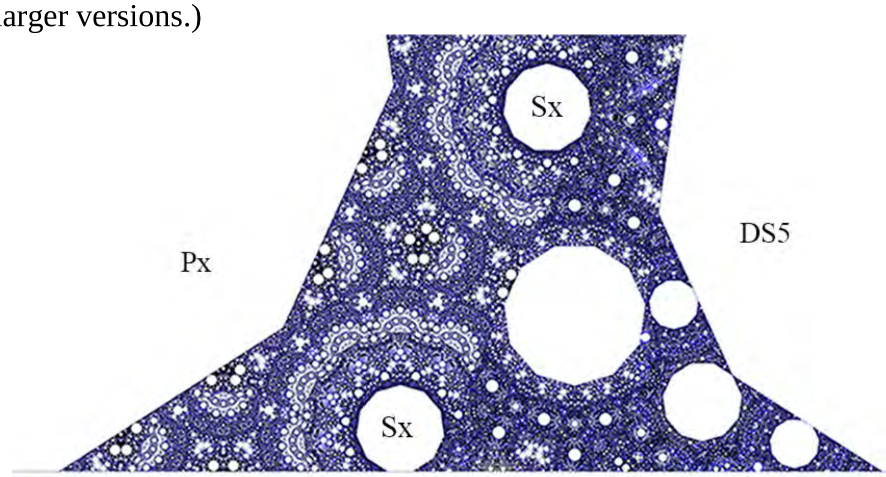
$$-\left(\left(2\left(11i+2(-1)^{1/22}-5(-1)^{3/22}-5(-1)^{5/22}+2(-1)^{7/22}-3(-1)^{9/22}-2(-1)^{13/22}+2(-1)^{17/22}-11(-1)^{19/22}+3(-1)^{21/22}\right)\right)/\right.$$

$$\left.\left(\left(5-(-1)^{1/11}+6(-1)^{2/11}-(-1)^{3/11}+5(-1)^{4/11}-3(-1)^{5/11}+4(-1)^{6/11}-4(-1)^{7/11}+4(-1)^{8/11}-4(-1)^{9/11}+3(-1)^{10/11}\right)\left(\cot\left[\frac{3\pi}{22}\right]+\cot\left[\frac{5\pi}{22}\right]\right)\right)\right)$$

The ratio hSx/hN will be in the scaling field S_{11} which is generated by x = GenScale[11] = Tan[π/11]·Tan[π/22]. **AlgebraicNumberPolynomial[ToNumberField[hSx/hN, GenScale[11]]** yields p(x) = $-\frac{9}{4}+52x+\frac{63x^2}{2}+x^3-\frac{5x^4}{4}$. This is what we call the 'characteristic' polynomial for hSx. It will be unchanged for any other hN and can also be used for N = 22 since S_{22}= S_{11} and the S[9] tile is a surrogate N = 11 with known height. Here inside N = 11, hSx = p(x)·1 ≈ 0.0014424.

To construct an exact Sx using p_1 as reference: (i) MidPointSx = {p_1 + hSx*Tan[4π/11], -1} ≈ {-6.4628351341351591344, -1} (ii) cSx = MidPointSx + {0,hSx}
(iii) rSx = RadiusFromHeight[hSx,11] (iv) Sx = RotateVertex[cSx + {0,rSx},11,cSx]

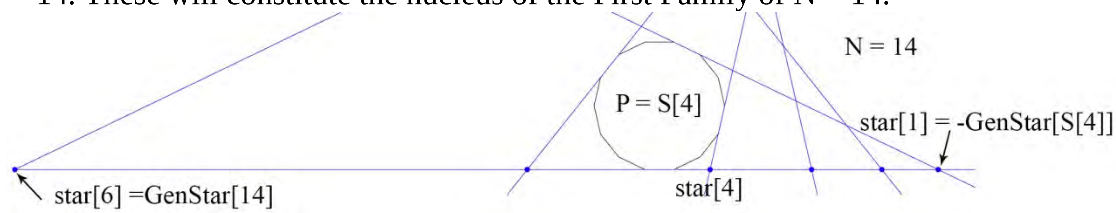
Under the outer- billiards map τ, the edges of tiles such as Sx are part of the singularity set W – also known as the 'web'. This set will be defined in Section 4. For a regular N-gon it can be obtained by mapping the extended edges of N under τ or τ^{-1}. Below is a portion of W in the vicinity of Sx. Note that Sx has a clone obtained by rotation about the center of Px. In the limit this web is probably multi-fractal. See Example 5.3 and [H6]. (Click on the main image or Sx to download larger versions.)



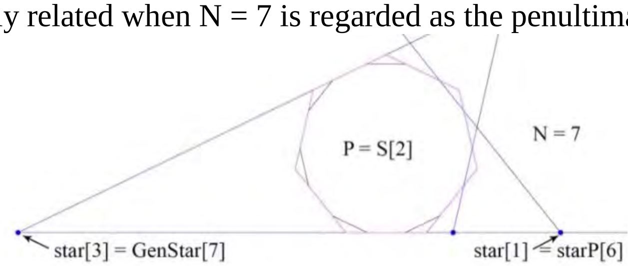
It is our contention that all the polygonal 'tiles' (regular or not) which arise from the outer billiards map of N are defined by scales which lie in the scaling field of N. For N = 11, knowing the exact parameters of 'third-generation' tiles like Sx allow us to probe deeper into the small-scale structure of N = 11 – which is almost a total mystery. But algebraic results like this point to the fact that each new 'generation' will tend to involve a significant increase in the complexity of any algebraic analysis. Therefore it may be impossible at this time to probe deeper than 10 or 12 generations for N = 11. Since the generations scale by GenScale[11] = Tan[π/11]·Tan[π/22] ≈ 0.0422171, the 25th generation would be on the order of the Plank scale of $1.6\cdot10^{-35}$ m. In the words of R. Schwartz, "*A case like N = 11 may be beyond the reach of current technology*."

Section 2. Conforming Regular Polygons

The S[4] tile from Example 1.1 is 'conforming' to the bounds of the star polygon of N =14 because they share the same 'base' edge and the right-side GenStar of S[4] is star[1] of N as shown below. Clearly there are an infinite number of such conforming N-gons for N = 14, but S[4] also shares star[4] (and star[5]) of N and there are only 6 such 'strongly conforming' tiles for N = 14. These will constitute the nucleus of the First Family of N = 14.



When N is odd, the conforming tiles are again relative to the star[1] point as shown below for N = 7. These conforming tiles must be 2N-gons so that they can share their penultimate star point with N. Note that P here cannot be replaced with its matching magenta heptagon and that applies to S[4] above, but any odd S[k] for N = 14 would be 'androgynous'. Once again the strongly conforming tiles like S[2] will be the nucleus of the First Family of N = 7. (The families of N = 14 and N = 7 are closely related when N = 7 is regarded as the penultimate S[5] of N = 14.)



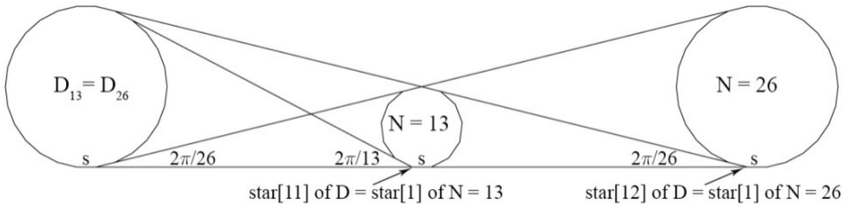
Definition 2.1 (i) If N is even and P is a regular N-gon or N/2-gon, P is *conforming* relative to N if P shares the same base edge as N and GenStar of P is ± star[1] of N. (ii) If N is odd and P is a regular 2N-gon then P is *conforming* relative to N if P shares the same base edge as N and star[N-2] of P is ± star[1] of N. In both cases P is said to be *strongly conforming* if it is conforming and also shares another star point with N.

Lemma 2.1 (Conformal Replication) Every regular N-gon has a strongly conforming D_N tile which is identical to N for N even and a regular 2N-gon with same side as N for N odd.

Proof: Set -starD_N[1] = GenStar[N] and center offset -sN/2. When N is even this center and side length defines a regular N-gon D_N which is identical to N. By the reflective symmetry of N, D_N

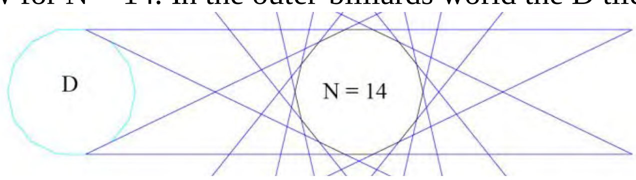
must have -GenStar[D_N] equal to star[1] of N. When N is odd, use GenStar[N] and the same offset of -sN/2 to construct a regular 2N-gon D_N. Because the exterior angle of D_N is half of the exterior angle of N, D_N will have star[N-2] equal to star[1] of N so it is strongly conforming. □

Example 2.1 When N is odd, the matching D_N is a 2N-gon, so it can also serve as the D for a 2N-gon, as shown here for N = 13 and N = 26.



This means that N = 13 will be a natural part of the First Family of N = 26, and we will call them an 'M-D' pair (Definition 2.3). Since sN = sD, hN/hD = scale[2] of D which is also SC(D).

It should be clear that D_N will be the largest possible strongly conforming tile relative to N as shown below for N = 14. In the outer-billiards world the D tiles are globally maximal.

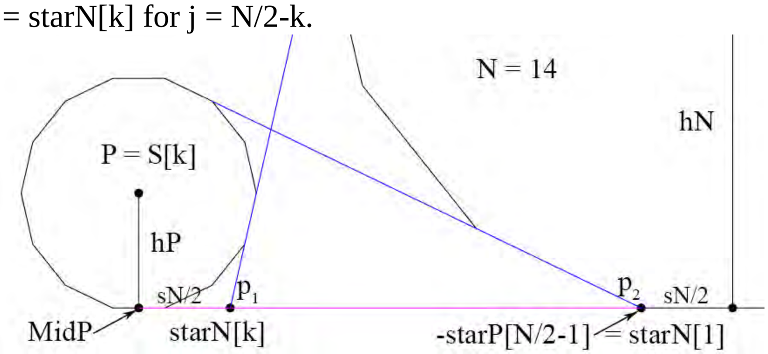


Corollary 2.1 For a regular N-gon, there is a conforming regular N-gon or 2N-gon P with hP $\leq$ hD_N.

Proof. By the Conformal Replication Lemma, there is a conformal D_M for any regular N-gon M embedded in N which shares the base edge and star[1] of N. This D_M will also be conformal relative to N because N and M have the same exterior angle. Therefore a conforming D_M must exist for any $hD_M \leq hD_N$. □

Theorem 2.1 (First Family) For a regular N-gon every star[k] point defines a unique S[k] tile which is strongly conforming and has horizontal center displacement -sN/2 relative to star[k].

Proof: Suppose that N is even with p_1 = starN[k]. Let P be the conforming regular N-gon with center displacement -sN/2 relative to p_1 as shown here. Such a P must exist by Corollary 2.1 of the Conformal Replication Lemma. We will show that P must have star[k] as a star point and -starP[j] = starN[k] for j = N/2-k.



This graphic matches Example 1.1, so k = 4 and j = 3. Every conforming P has -starP[N/2-1] = star[1] of N, so they all share a global 'index' of N/2-1, and here we claim that -starP[N/2-k] = star[k] of N, so the 'local' index of P is N/2-k. When k = 1 this local index matches the global index but the diagram above is still valid with $p_1 = p_2$. On the other extreme, when k = N/2-1, the local index is 1 and P is the D_N from the Replication Lemma. (This 'retrograde' form of the local index is due to the numbering convention of the S[k].)

The magenta displacement from p_2 to MidP can be written from the two different perspectives:
(a) Relative to N, it is the same as the displacement of star[k], namely -hN·Tan[kπ/N].
(b) Relative to P and starN[1], this displacement is -hP·Tan[(N/2-1)π/N].

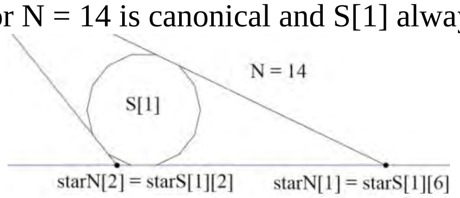
Therefore hN·Tan[kπ/N] = hP·Tan[(N/2-1)π/N], so $\frac{hP}{hN}=\frac{Tan[k\pi/N]}{Tan[(N/2-1)\pi/N]}=\frac{s_k}{s_{N/2-1}}=\frac{s_1}{s_{N/2-k}}$

This last equality follows from the fact that for N even, $s_{N/2-k} = 1/s_k$ as noted in Lemma 1.2.

Therefore $\frac{hS[k]}{hN}=\frac{s_1}{s_{N/2-k}}$ and $hS[k]\cdot s_{N/2-k}$ = hN·s1. As in Example 1.1, hN·s1 is the horizontal displacement of MidN from –star[1] of N, which is –sN/2 , and $hS[k]\cdot s_{N/2-k}$ is the horizontal displacement of MidS[k] from star[N/2-k] of S[k] (which is star[k] of N). Therefore the displacement of MidS[k] from star[k] is –sN/2 as expected.

For k > 1 all the S[k] will be strongly conforming because they share two distinct star points with N. When k = 1 the local index is the same as the global index but the calculations above are valid when $p_1 = p_2$, so $\frac{hS[1]}{hN}=\frac{Tan[\pi/N]}{Tan[(N/2-1)\pi/N]}=\frac{s_1}{s_{N/2-1}}=s_1^2$. (Recall that this is GenScale[N])

Therefore S[1] can be constructed without the help of star[2], but we will show that the illustration below for N = 14 is canonical and S[1] always has index 2 relative to star[2].



To see this, apply the Two-Star Lemma with opposite sides and (hypothetical) local index 2.

This yields $\frac{hS[1]}{hN}=\frac{Tan[2\pi/N]-Tan[\pi/N]}{Tan[(N/2-1)\pi/N]+Tan[2\pi/N]}=Tan[\pi/N]^2$ as above.

Therefore for N even all of the S[k] tiles are strongly conforming and share the same offset.

The N-odd case can be done in the same fashion, but it can also be derived from the N-even case because the D tile of N will be a 2N-gon that shares its penultimate star point with N - so N will be DS[N-2] of D and we can use the DS[k] of D to scale the S[k] of N. As in the Scaling Lemma, the scales of any N-gon and 2N-gon have a trivial relationship since by definition s_k of N is s_{2k} of 2N. Therefore for N-odd, the S[k] of N will be congruent to S[2k] of D, and by the even-case:

$\frac{hS[k]}{hN}=\frac{hS[2k]}{h2N}$ of 2N = $\frac{s_2}{s_{2N/2-2k}}=\frac{s_2}{s_{N-2k}}$ of 2N so this is a simple doubling of the indices.

Therefore, $hS[k]\cdot s_{N-2k} = h2N\cdot s_2$ – which is the horizontal displacement of -star[2] of 2N so it is -star[1] of N = -sN/2. The left side says that the local index of S[k] is N-2k and this matches star[2k] of 2N – which is star[k] of N. So the S[k] have displacement sN/2 as desired. □

Therefore the N-odd case is a simple doubling of the indices from the N-even case and these cases are related by a 'gender change' between 2N and N - mediated by scale[2] of 2N. In summary the following scales will guarantee that all S[k] are strongly conforming to N.

For N even: $\dfrac{hS[k]}{hN}=\dfrac{s_1}{s_{N/2-k}}=s_1\cdot s_k$ of N - which is scale[N/2-k] of N

For N odd: $\dfrac{hS[k]}{hN}=\dfrac{s_2}{s_{N-2k}}=s_2\cdot s_{2k}$ of 2N - which is scale[2(N/2-k)]/scale[2] of 2N

We will show in Lemma 4.1 that for all N-gons, cS[k] and star[k] are height/radius duals - so if star[k] is a midpoint of some N-gon P then cS[k] is the adjacent vertex of P. When N is even the most important scaling is hS[1]/hN = s_1^2 = GenScale[N] but when N is odd, hS[1]/hN = s_2^2 which is a gender mismatch and not a very useful scale. This can be corrected by using DS[1] instead. When N is odd, S[1] is always a 2N-gon but DS[1] (and all odd DS[k]) can be either N-gons or 2N-gons without violating strong conformity, so DS[1]/hN is not a gender mismatch and is equal to hDS[1]/(hD·scale[2]) = GenScale[2N]/scale[2] = $s_1\cdot s_2$ of 2N = GenScale[N].

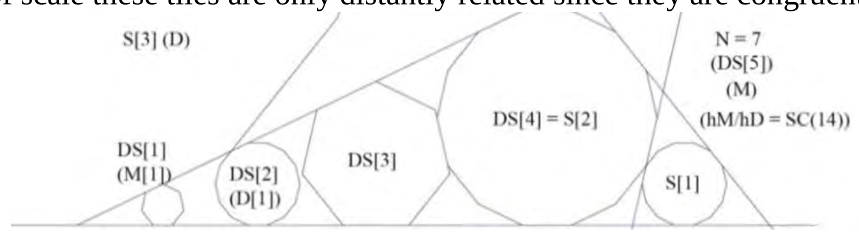
When N is twice-odd, both of these GenScale options are possible: hDS[1]/hD = hS[1]/hN = GenScale[N] or hDS[1]/h(N/2) = GenScale[N/2]. If DS[1] is indeed an N/2-gon, GenScale[N] is a gender mismatch and typically not an algebraic integer , while GenScale[N/2] = $s_1\cdot s_2$ of N = Tan[π/N]·Tan[2π/N] is always an algebraic integer and a unit. (When N is twice-even GenScale[N] is not a gender mismatch since DS[1] is an N-gon, so it it an algebraic unit.)

Lemma 2.2 (First Family Scaling) For all regular N-gons, hS[1]/hS[k] = scale[k] of N.

Proof: For N-even hS[1]/hN = s_1^2 and hS[k]/hN = $s_1/s_{N/2-k} = s_1\cdot s_k$ so hS[1]/hS[k] = s_1/s_k of N. For N-odd hS[1]/hS[k] = hS[2]/hS[2k] of 2N = hS[2]/hS[1] /hS[2k]/hS[1] = $(s_2/s_1)/(s_{2k}/s_1)$ = scale[2k]/scale[2] of 2N = scale[k] of N by Lemma 1.1. □

This means that all the S[k] can be scaled relative to S[1] but it is also necessary to know how S[1] scales with respect to N. As noted above, for N even, hS[1]/hN is GenScale[N] so S[1] can serve as a 2nd generation N. When N is odd, it is natural to shift the emphasis to D since hDS[1]/hN = GenScale[N]. Therefore DS[1] can serve as a 2nd generation N. See Example 2.2.

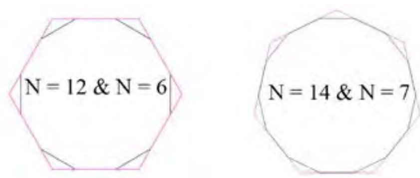
Example 2.2 (The First Family of N = 7) For N odd, D is S[<N/2>] which is S[3] here. This means that N will be the penultimate tile of D, namely DS[N -2], so N is called the M-tile of D. (See Definition 2.3 below). By the First Family Theorem, hN/hD = scale[2] of D = SC(D) so as far as scaling is concerned, N and D just differ by gender. DS[1] and DS[2] have this same scale[2] relationship and that is why they are known as D[1] and M[1]. Therefore hDS[1]/hN = hM[1]/hM = GenScale[N] and DS[1] is a candidate for a 2nd generation N = 7. Lemma 2.2 also says that hS[1]/hS[2] is scale[2] of N, but N is odd so scale[2] is longer a gender change and in terms of scale these tiles are only distantly related since they are congruent to DS[2] and DS[4].



Whenver N is odd, these M-D relationships must hold, but this will contradict the First Family Theorem unless tiles such as DS[1] and DS[N-2] can be regarded as N-gons instead of 2N-gons. We show below that for N odd, all odd DS[k] can be replaced with their N-gon 'gender-duals' without violating the First Family Theorem, so they are androgynous tiles. (When the initial N is even, D is a clone of N so the odd case does not arise.) The Twice-odd Lemma (Lemma 4.2) gives the transformation that relates the First Families of N = 7 and N = 14. Since these two have equivalent cyclotomic fields, the First Families should be algebraically equivalent - but this equivalence only makes sense if the odd DS[k] are replaced with their gender duals. This will help to restore some of the even-odd parity of the S[k] and now it makes sense that DS[1] could be regarded as a 2^{nd} generation N = 7. We will explain in Secion 4 why this gender change occurs in the web of the outer-billiards map – and why this issue is important in the dynamics of τ or any piecewise isometry based on rational rotations. In physics it is known as the 'parity' issue between parameters or particles which act like scalers or vectors.

Definition 2.2 For N even, a regular N/2- gon is the *outer-dual* of N if the N/2-gon is circumscribed about the N-gon with matching 'base' edge, center and height. In this case sN/s(N/2) will be SC(N) = s_1/s_2 of N. When N is twice-odd, this outer-dual will be called the *parity-dual* or *gender-dual* of N. This the only case where GenStar[N] = GenStar[N/2].

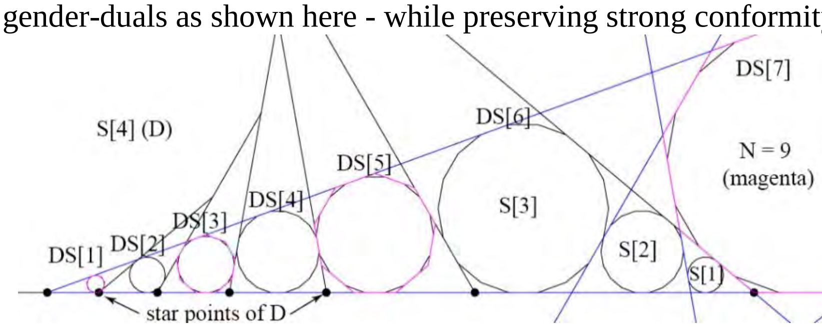
Example 2.3 Below are the outer-duals for N = 12 and N = 14.



Lemma 2.3 When N is odd , the First Family Theorem implies that the even S[k] of D_N are congruent to the S[k/2] of N. Here we show that the odd S[k] of D_N can be replaced by their gender duals and the resulting N-gons are still strongly conforming to D.

Proof: Suppose P is the gender-dual of an S[k] of D. P will an N-gon with the same center and height as S[k] and GenStar[P] will be GenStar[S[k]] iff N is odd. Therefore P will be conforming. In addition star[j] of P will be star[2j] of S[k]. By the First Family Theorem, the local index of S[k] of N is N-k and this will be even iff k is odd, so P will have new local index (N-k)/2 and hence be strongly conforming. □

Example 2.4 (The First Family of N = 9) As with N = 7 above, this will include both the S[k] of N and the (right-side) DS[k] of D. Note that S[1], S[2] and S[3] are congruent to DS[2], DS[4] and DS[6] and by Lemma 2.3, DS[1], DS[3], DS[5] and DS[7] can be swapped with their magenta gender-duals as shown here - while preserving strong conformity with D (and N = 18).

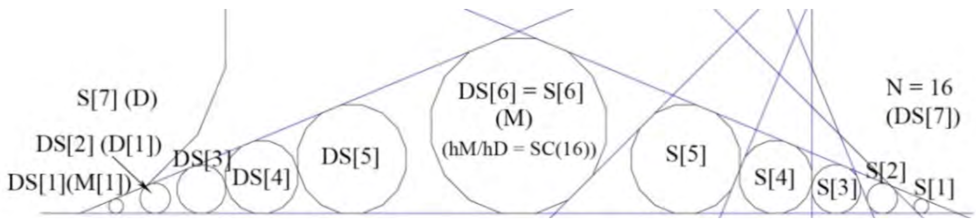


Since N = 9 is the penultimate tile of D, it is also called M. As with N = 7, DS[1] and DS[2] are scaled copies of M and D so they are known as M[1] and D[1].

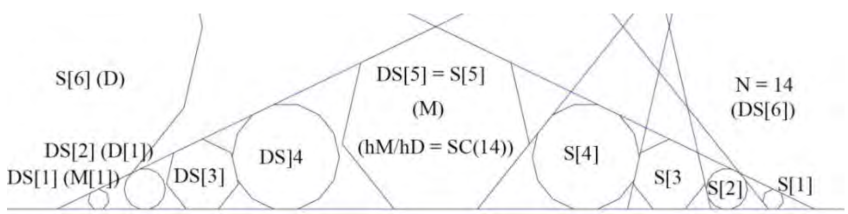
Definition 2.3 (M tiles) If P is a regular N-gon with N >4 even, the ‘penultimate’ S[N/2-2] of P is called the *M-tile* of P. (Since hM/hP = scale[2] of P, the scales of P and M will be equivalent under scale[2]. When P is twice-odd, scale[2] is called SC(P) and M can be an N/2 gon.)

Definition 2.4 (First Family) For any regular N-gon, the First Family Theorem defines the strongly conforming S[k] tiles for $1 \leq k \leq$ <N/2>. These tiles will be called the First Family Nucleus and S[<N/2>] will be D_N (a.k.a. D). Based on the definition above, the penultimate tile of D will be called M and for N >4, DS[1] and DS[2] will be called M[1] and D[1].

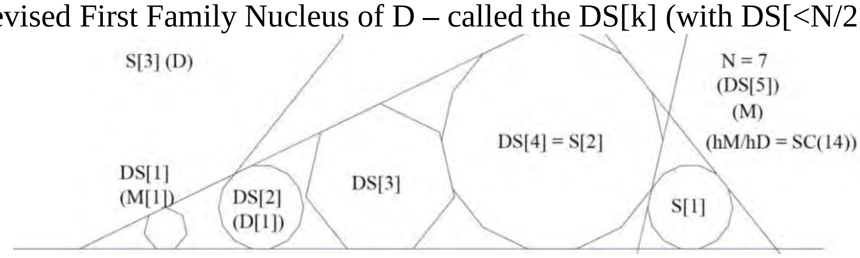
(i) For N twice-even, the First Family will consist of the S[k] in the First Family Nucleus together with the (right-side) First Family Nucleus of D – called the DS[k].



(ii) For N twice-odd, the odd S[k] in the First Family Nucleus will be replaced with their N/2 counterparts, and the First Family will consist of this revised First Family Nucleus together with the (right-side) revised First Family nucleus of D – called the DS[k].



(iii) For N odd, the First Family will consist of the First Family Nucleus together with the (right-side) revised First Family Nucleus of D – called the DS[k] (with DS[<N/2>] usually omitted).



Since the M tile of N = 14 can be regarded as a scaled copy of N = 7, case (iii) is essentially half of case (ii) and we will make this relationship precise in the Twice-odd Lemma of Section 4. As indicated in the introduction, most of the algebraic and geometric complexity of a regular N-gon can be traced to the cyclotomic field $\mathbb{Q}_N$, which is defined below. For N twice-odd, $\mathbb{Q}_N = \mathbb{Q}_{N/2}$, so the scaling and singularity sets of N = 11 and N = 22 are interchangeable and there are theoretical and computational advantages to regarding N = 11 as embedded in N = 22.

For example, the extra symmetry of N-even allows the outer-billiards map to be replaced with the simpler toral Digital Filter map – as explained in the Appendix. The singularity set local to Sx in Example 1.2 was generated in this fashion – but it still required billions of iterations.

Section 3. Evolution of First Families - Generation Scaling

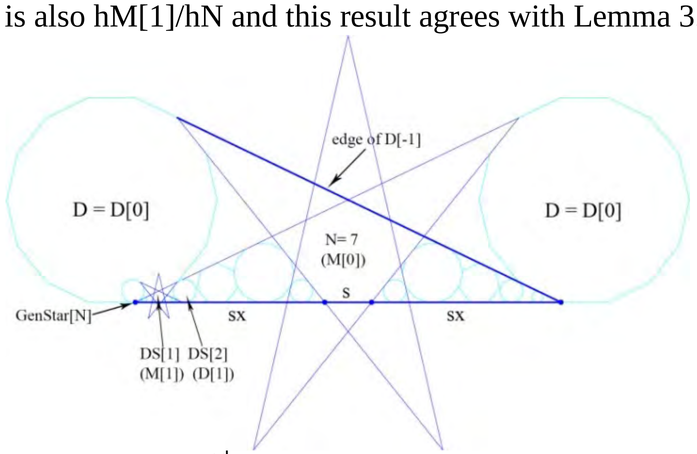
Since the First Family tiles are regular polygons, they have well-defined secondary families and this is an invitation to recursion. For the quadratic polygons N = 5,8,10 and 12, these secondary families appear to exist on all scales under the outer-billiards map τ and the resulting topology is fractal with geometric scale given by GenScale[N] – as described in Sections 4 and 5. We will show later in this section that for all N-gons, GenScale[N] (or GenScale[N/2]) generates the scaling field S_N so it should play an important role in the topology of τ. Here we will present evidence that this GenScale scaling is actually inherent in the star polygons of N – and hence would be expected to reflect the geometry and dynamics of τ. The following Lemma relates the two different expressions for GenScale[N].

Lemma 3.1 For all regular N-gons, hDS[1]/hN = GenScale[N]

Proof: DS[1] is S[1] of D, so when N is even hDS[1]/hN = hS[1]/hD = hS[1]/hS[N/2-1] and this is scale[N/2-1] which is GenScale[N]. When N is odd, D is twice-odd so hDS[1]/hD = GenScale[D] as above, but this is a gender-mismatch which can be corrected by dividing both sides by scale[2] of D so hDS[1]/hN = GenScale[D]/scale[2] of D = GenScale[N]. □

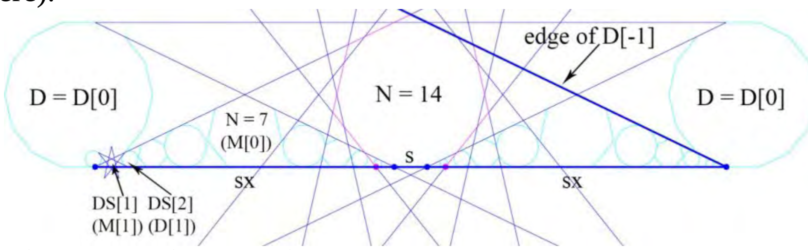
Based on this Lemma, the S[1] tile of D will typically play the part of the 'next generation' N. This is called 'generation' scaling – and theoretically it could be continued to generate infinite (ideal) sequences of generations converging to GenStar[N] as in Definition 3.3 to follow. To show that this DS[1] scaling is inherent in the star polygons of N, there are 3 cases to consider.

(i) (N odd) Note that N = 7 can be regarded as the S[1] tile of a 'parent' D tile called D[-1]. One edge of D[-1] is shown here in dark blue. The scaling between DS[1] (M[1]) and M[0] must be the same as the scaling between the parents D[0] and D[-1]. Here sD = sN so sD[0]/sD[-1] = sN/sD[-1] = s/(2sx+s) = Tan[π/7]·Tan[π/14] = GenScale[7] = sM[1]/sN. Since the genders match, this ratio is also hM[1]/hN and this result agrees with Lemma 3.1.



This is what we call the (ideal) 2nd generation of N = 7 with M[1] and D[1] serving as N and D. The kth generaton with M[k-1] and D[k-1] would be scaled by $GenScale[7]^{k-1}$.

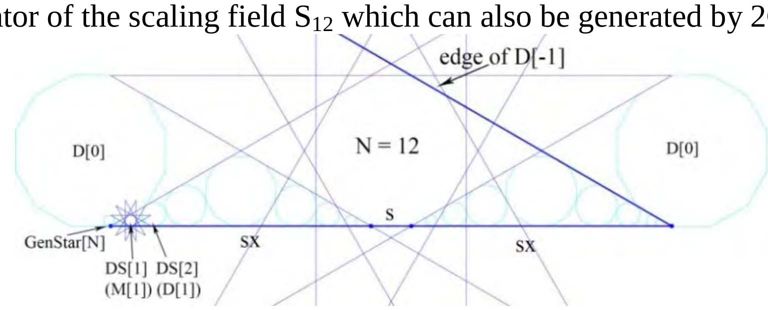
(ii) (N twice-odd) This should be equivalent to case(i) above, but now N =14 is at the origin playing the role of S[1] of the large D[-1]. Therefore sD[0]/sD[-1] = s/(2sx+s) = $\text{Tan}^2[\pi/14]$.
= GenScale[14]. This ratio is indeed sM[1]/sN as above, but now it is a gender mismatch, and not even an algebraic integer. This can be rectified by dividing both sides by scale[2] of N =14, to get sM[1]/sM[0] = hM[1]/hM[0] = GenScale[14]/scale[2] = GenScale[7]. Geometrically this is the same as replacing N = 14 with it magenta gender-dual (and scaled-up D[0]′ which is omitted here).



Now sD[0]′/sD[-1] = s′/(2sy + s′) where sy = star[2][[1]]-GenStar[[1]] and s′ = -2star[2][[1]] (magenta). This yields scale[6]/scale[2] which is GenScale[7] = hM[1]/hM[0] as in case(i).

Therefore N = 14 has scaling compatible with N = 7 as long as the S[1] tiles on the edges of N = 14 are heptagons. Algebraically this compatibility is driven by the equivalence of their cyclotomic fields under $\zeta \rightarrow -\zeta$ where $\zeta = \exp(2\pi i/N)$. The outer-billiards map and all rational piecewise mappings would be expected to preserve this equivalence.

(iii) (N twice-even) This resembles case(ii) but now there is no gender mismatch and N = 12 is the correct S[1] of D[-1]. By definition sD[0]/sD[-1] = s/(2sx+s) = $\text{Tan}^2[\pi/12]$ = GenScale[12] . This is also sS[1]/sN = hS[1]/hN = $7- 4\sqrt{3}$ – which is an algebraic unit (with inverse is $7+ 4\sqrt{3}$). It is a generator of the scaling field S_{12} which can also be generated by $2\text{Cos}[2\pi/12] = \sqrt{3}$.



As in case(ii), D[0] is congruent to N so DS[1] and DS[2] have the same scaling as S[1] and S[2] of N. S[1] will still be the M[1] 'penultimate' tile in the First Family of S[2], but S[2] cannot be in the First Family of S[1] since they have the same gender. This means that the 2^{nd} generation on the edges of N will be dominated by the family of S[1] – not S[2]. In Example 5.6 we will see that the dominant tiles in the 2^{nd} generation of N = 12 are the M[k], not the D[k]. For any twice-even N-gon of the form 8k+4, this should be a similar issue – while the 8k cases would be expected to have 2^{nd} generations with both S[1] families and S[2] families at D[1].

The singularity set W of the outer-billiards map will be defined in Section 4. It is formed by iterating the extended edges of N – so it is based on the star polygons of N. The examples here show that generation scaling is an intrinsic part of the star polygon geometry for a regular N-gon, so it would be expected that W shares this symmetry – and at least allows for the possibility of self-similar generations to exist at GenStar. This is part of the 4k+1 conjecture of Section 4.

Note: The results above show that any even N-gon can support these ideal families on its edges. When N is odd this is no longer true, and this why it is necessary rely on the matching D tile to support these families. The problem with N-odd is that the S[1] and S[2] tiles are now 2N-gons, so their evolution is not always compatible with N. S[1] and S[2] are now congruent to DS[2] and DS[4] so they are only distantly related, and the First Family of S[2] will typically not contain S[1]. Since the web of S[2] will always generate S[1] on the edges of N, these tiles are still related – but their relationship is not a simple First Family issue. The Edge Conjecture of Section 4 attempts to classify the types of dynamics that can occur on the edges of any N-gon, and for N-odd these dynamics tend to be much more complex that the N-even case.

The cyclotomic field of N

Definition 3.1 For a regular N-gon, the cyclotomic field $\mathbb{Q}_N$, is the algebraic number field which can be generated by $\zeta = \exp(2\pi i/N) = \cos(2\pi/N) + i\sin(2\pi/N)$. As a vector space $\mathbb{Q}_N$ is the direct sum of its real and imaginary parts, $\mathbb{Q}_N^+$ and $\mathbb{Q}_N^-$, Since $\zeta + \zeta^{-1} = 2\cos(2\pi/N)$, $\mathbb{Q}_N^+$ can be generated by $\lambda_N = 2\cos(2\pi/N)$. Since ζ has (minimal) degree $\varphi(N)$ and complex conjugation in $\mathbb{Q}_N$ is always an automorphism of order 2, $\mathbb{Q}_N^+$ has degree $\varphi(N)/2$. It is called the maximal totally real subfield of $\mathbb{Q}_N$. The full field $\mathbb{Q}_N$ is always a quadratic extension of $\mathbb{Q}_N^+$.

Because of the Lemma below $\mathbb{Q}_N^+$ is also called the *'scaling field'* of N – written S_N. Since $\mathbb{Q}_N$ is equivalent to $\mathbb{Q}_{N/2}$ for N twice-odd, $S_N = S_{N/2}$.

Lemma 3.2 (Scaling Field Lemma) For any regular N-gon, the maximal real subfield $\mathbb{Q}_N^+$ has a unit basis consisting of the primitive (canonical) scales.

Proof: For a given N we have defined $s_k = \tan(k\pi/N)$ for $1 \leq k < N/2$. These 'star points' are classified as 'primitive' if (k,N) = 1 and 'degenerate' otherwise. Here we show that the set of primitive scales, $T = \{t_K = s_1/s_k: (k,N) = 1\}$ is a basis for $\mathbb{Q}_N^+ = \mathbb{Q}_N \cap \mathbb{R}$. $\mathbb{Q}_N$ can be generated by any 'primitive' N-th root of unity of the form $\zeta^k = \exp(2k\pi i/N)$ with (k,N) = 1 – so the indices of the primitive scales are also indices of primitive roots of unity.

Since $i\tan(\theta) = (e^{i2\theta}-1)/(e^{i2\theta}+1)$, $is_k = (e^{2k\pi i/N}-1)/(e^{2k\pi i/N}+1) = (\zeta^k-1)/(\zeta^k+1)$ so is_k is in $\mathbb{Q}_N$ and scale[k] = $\tan[\pi/N]/\tan[k\pi/N] = is_1/is_k = \left[\frac{\zeta-1}{\zeta+1}\right]\left[\frac{\zeta^k+1}{\zeta^k-1}\right]$ is in $\mathbb{Q}_N^+$. We will show that when (k,N) = 1, scale[k] is an algebraic integer in $\mathbb{Z}[\zeta]$ and its inverse coscale[k] = $\cot[\pi/N]/\cot[k\pi/N]$ is also an integer. Regrouping terms, scale[k] = $\left[\frac{\zeta-1}{\zeta^k-1}\right]\left[\frac{\zeta^k+1}{\zeta+1}\right]$. The term on the left is called a 'cyclotomic unit' in $\mathbb{Z}[\zeta]$, but we will show from first principles that this product is a unit. Since (k,N) = 1, there is a (rational) integer j such that kj=1(ModN). Therefore

$$\left[\frac{\zeta-1}{\zeta^k-1}\right] = \left[\frac{(\zeta^k)^j-1}{(\zeta^k)-1}\right] = \sum_0^{j-1}(\zeta^k)^i \text{ and when k is odd } \left[\frac{\zeta^k+1}{\zeta+1}\right] = \left[\frac{(-\zeta)^k-1}{(-\zeta)-1}\right] = \sum_0^{k-1}(-\zeta)^i$$

If k is even N must be odd, so repeat the above with $k' = k+N$. This substitution leaves scale[k] and coscale[k] unchanged. Therefore scale[k] is an algebraic integer.

For coscale[k] replace ζ with -ζ so when k is odd the two quotients are $\sum_{0}^{j-1}(-\zeta^{k})^{i}$ and $\sum_{0}^{k-1}\zeta^{i}$.

When k is even, again replace k with k′ = k+N which will be odd, and the inverse j can be chosen odd so that it will preserve the sign change. Therefore scale[k] is an algebraic unit in $\mathbb{Q}_N^{+}$ with inverse coscale[k].

To show that the set T of primitive scales forms a basis for $\mathbb{Q}_N^{+}$, note that |T| = φ(N)/2 because (k,N) = 1 implies that (N-k, N) = 1. It only remains to show that the primitive scales are independent over $\mathbb{Q}$.

Suppose that $\sum_{1\le k<N/2} a_i t_k = 0$ with (k,N) =1 and $a_i \in\mathbb{Q}$, then $\frac{1}{s_1}\sum_{1\le k<N/2} a_i t_k = \sum_{1\le k<N/2} a_i r_k = 0$ where r_k = cot[kπ/N] with (k,N) = 1. This contradicts the Siegel-Chowla result that the primitive r_k are independent over $\mathbb{Q}$. Therefore the primitive scales are a unit basis for $\mathbb{Q}_N^{+}$. □

Example 3.1 (N = 11) Set $\zeta = \zeta_{11}$ then scale[4] = $\frac{\mathrm{Tan}[\pi/11]}{\mathrm{Tan}[4\pi/11]} = \left[\frac{\zeta-1}{\zeta+1}\right]\left[\frac{\zeta^4+1}{\zeta^4-1}\right] = \left[\frac{\zeta-1}{\zeta^{15}-1}\right]\left[\frac{\zeta^{15}+1}{\zeta+1}\right]$

$$= \left[\frac{(\zeta^{15})^3-1)}{\zeta^{15}-1}\right]\left[\frac{(-\zeta)^{15}-1}{-\zeta-1}\right] = \sum_{j=0}^{2}(\zeta^{15})^j\cdot\sum_{j=0}^{14}(-\zeta)^j = (1+\zeta^4+\zeta^8)\cdot\sum_{j=0}^{14}(-\zeta)^j$$

coscale[4] = $\frac{Cot[\pi/11]}{Cot[4\pi/11]} = \sum_{j=0}^{2}(-\zeta^{15})^j\cdot\sum_{j=0}^{14}\zeta^j = (1-\zeta^4+\zeta^8)\cdot\sum_{j=0}^{14}\zeta^j$

This shows a relationship between scale[4] and ζ^4 -which goes back to star[4] and S[4]. Since ζ^4 is a generator of $\mathbb{Q}_{11}$ this relationship can be made exact.

AlgebraicNumberPolynomial[ToNumberField[scale[4], Exp[2*Pi*I/11]^4]],x] yields $-1+2[x^2+x^3-x^5-x^6+x^8+x^9]$**.** Since $S_{11} = \mathbb{Q}_{11}^{+}$ is generated by λ_{11} = 2cos(2π/11) the integers in the field $\mathbb{Q}[\lambda_{11}]$ must also be in the ring $\mathbb{Z}[\lambda_{11}]$, where scale[4] simplifies to $3-4x-2x^2+2x^3$**.**

Of course the simplest representation of scale[4] is just x since scale[4] (or any primitive scale) can be used as an alternative generator for S_{11} - but we will prefer to use scale[5] = GenScale[11] which has the form $1-6x^2+2x^4$. Among the scales of N = 11, this will be the closest algebraic relative to λ_{11} and in general we have seen that scaling polynomials in x = GenScale[N] may be meaningful. To use GenScale[N] as a generator for $S_N = \mathbb{Q}_N^{+}$, it is sufficient to show that it is a primitive scale for N (or show that it is in S_N and has minimal degree φ(N)/2.)

Lemma 3.3 In Lemma 1.2 we showed that GenScale[N] is Tan^2[π/N] for N even and Tan[π/N]·Tan[π/2N] for N odd. Here we show that GenScale[N] is a primitive scale except when N is twice-odd – in which case GenScale[N/2] is primitive.

Proof: By definition GenScale[N] = scale[<N/2>] which is primitive iff (<N/2>,N) = 1, This is true iff N-1< (2<N/2>) < N. Since (N,N+1) =1, (<N/2>,N) is either 1 or 2. When N is odd, <N/2> = (N-1)/2 is even so (<N/2>,N) = 1. When N = 2M, <N/2> = M-1 so (<N/2>,N) = (M-1,2M) which is only 1 when N is twice-even. □

Corollory 3.1 For a regular N-gon, GenScale[N] is a unit generator of the scaling field S_N except when N is twice-odd, in which case GenScale[N/2] is a unit generator of $S_{N/2}$ and S_N.

Algebraically Cos[θ] and Tan^2[θ] are related by the 'half-angle' formula of Diophantus.

$$\cos(2\theta)=\frac{1-\tan^2(\theta)}{1+\tan^2(\theta)}$$ so for N even, $\lambda_N = 2\frac{\text{1-GenScale[N]}}{\text{1+GenScale[N]}}$ and for N odd $\lambda_{2N}=\frac{2}{\text{GenScale[N]+1}}$

So it is no surprise that $s_1^2 = Tan^2[\pi/N]$ can always generate $\mathbb{Q}_N^+$ but we have already noted that this may not be a good choice when N is twice-odd. See Example 3.3 below.

Since all the is_j are in $\mathbb{Q}_N$, the degenerate scales are still in $\mathbb{Q}_N^+$ so any such scale will be a linear combination of the primitive scales. The same is true for the degenerate s_k themselves:

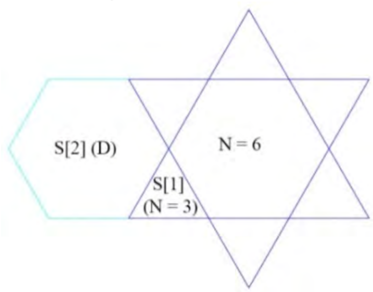
Corollary 3.2 For a given N, every s_j for j < N, is a linear combination of the primitive s_k.

Proof: Suppose (j,N) = m ≥ 1, then (j/m,N/m) = 1 and $s_{j/m}$ of N/m = s_j of N. Since $\mathbb{Q}_{N/m} \subseteq \mathbb{Q}_N$, $is_j \in \mathbb{Q}_N^-$ and the set $iS = \{is_k : (k,N) = 1, k < N/2\}$ forms a (vector space) basis for $\mathbb{Q}_N^-$ because $|iS| = \varphi(N)/2$ and the is_k are independent over $\mathbb{Q}$. Therefore is_j is a linear combination of the primitive is_k which implies that s_j is a linear combination of the s_k. □

Example 3.2 For a case like N = 18, known summation formula such as those found in [P] can be used to find the coefficients. For example tan(π/18) + tan(π/18 + π/3) + tan(π/18 + 2π/3) = tan(6π/18) = $\sqrt{3}$. Note that all the factors on the left side are primitive in N = 18, with k values 1, 7 and 13 (= -5), so tan(π/18) + tan(7π/18) - tan(5π/18) = $\sqrt{3}$. In general it can be very difficult to find these coefficients and the same applies to coefficients of the scales.

As noted above Tan^2[π/N] will always be a generator of S_N but when N is twice-odd it may not be a unit or even an integer. Therefore it would not be a good choice of generator. The First Family geometry shows clearly the gender issue discussed earlier in this section.

Example 3.3 (The First Family for N = 6) The First Family Theorem says that hS[1]/hN = scale[2] = GenScale[6] = Tan^2[π/6] = 1/3, but this is not an algebraic integer, so a better choice of scaling is hS[2]/hN= GenScale[3] = Tan[π/6]·Tan[π/3] = 1. This avoids the gender mismatch between N = 6 and S[1] and assigns the correct scaling to the web W.



For N = 3, 4 and 6, φ (N)/2 = 1 so they have linear algebraic complexity. This means that there will be no accumulation points in the outer-billiards web W because the web consists of rays or segments parallel to the sides of N and under the outer-billiards map, these segments are bounded apart by linear combinations of the vertices. When N is regular the vector space determined by the vertices has the same rank as $\mathbb{Q}_N$ - namely φ(N). Therefore the coordinate space of N-gons with linear complexity will have rank 2 over $\mathbb{Q}$ and hence affinely rational coordinates. Since the outer-billiards map is itself an affine transformation these 'rational' coordinates are preserved and the web is also affinely rational – with no limit points.

Definition 3.2 (Canonical polygons) Every regular N-gon defines a coordinate system, and any line segment or polygon P (convex or not) that exists in this co-ordinate system will be called *canonical* relative to N if for every side sP, the ratio sP/sN is in S_N. The Scaling Conjecture says that all tiles and line segments which arise in the web W of the outer-billiards map are canonical.

Example 3.4 Any line segment defined by a linear combination of star points of N is canonical because if T is such a linear combination T/s_1 is a linear combination of dual scales, so it is in S_N. Therefore every first generation S[k] web-based mutation is canonical because the edges are the difference of two star points. In addition every star polygon based on N is canonical. Our convention for scaling N-gons is to scale them relative to another N-gon and in this case side scaling is the same a height scaling. The only mixed-gender case is with N twice-odd and the matching M tile can be used as reference for scaling N/2-gons. All the scales of N/2 are also scales of N, so any N/2 scaling has an equivalent scaling using N.

Lemma 3.4 For a regular N-gon, the First Family S[k] tiles are canonical.

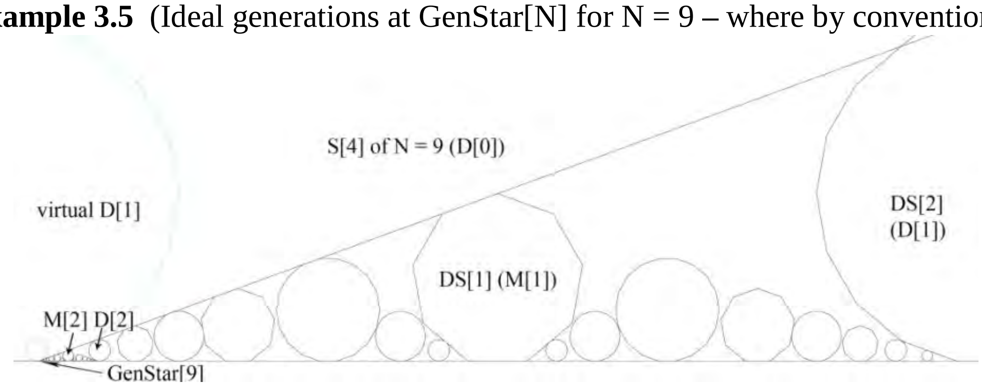
Proof: For N even, $\dfrac{hS[k]}{hN} = \dfrac{Tan[\pi/N]}{Tan[(N/2-k)\pi/N]}$ = scale[N/2-k] of N, which is in S_N.

For N odd $\dfrac{hS[k]}{hN} = \dfrac{Tan[2\pi/2N]}{Tan[(N-2k)\pi/2N]}$ = scale[N-2k]/scale[2] of 2N so it is in $S_{2N} = S_N$. □

Since the S[k] are canonical regular N-gons or N/2 gons, the First Families and subsequent families generated by the S[k] will also be canonical regular N-gons or N/2 gons. The linear or quadratic cases like N = 3,4,5,6, 8, 10, and 12 have webs consisting of only scaled First Family tiles with possible period-based mutations, so all tiles are canonical.

Definition 3.3 (Ideal generations at GenStar[N]**)** For any regular N-gon with N > 4 , define D[0] = D and M[0] = M (the penultimate tile of D). Then for any natural number k > 0, define M[k] and D[k] to be DS[1] and DS[2] of D[k-1] so for all k, M[k] will be the penultimate tile of D[k]. The (ideal) *kth generation* of N is defined to be the (ideal) First Family of D[k-1]. Therefore M[k] and D[k] will be 'matriarch' and 'patriarch' of the next generation – which is generation k+1. By Lemma 3.1, hM[k]/hM[k-1] will be GenScale[N] or GenScale[N/2] if N is twice-odd.

Example 3.5 (Ideal generations at GenStar[N] for N = 9 – where by convention M[0] = N)

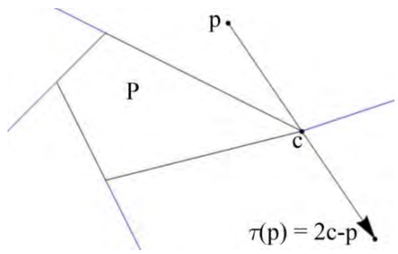


(The Mathematica code for these families is **FFM1 = TranslationTransform[cM[1]][First Family*GenScale]** and **FFM2 = TranslationTransform[cM[2]][FirstFamily*GenScale^2]**.)

Section 4. The Outer-Billiards map

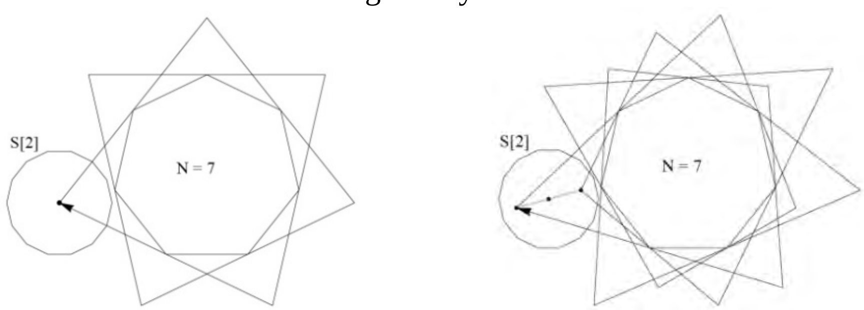
Except for motivation, the first three sections are independent of any mapping, and here we will show connections between the First Families and the dynamics of the outer-billiards map τ. Lemma 4.1 below shows that the center of each S[k] has a constant 'step-k' τ-orbit around N. These are the 'period- based resonances' that would be expected to occur for any rational piecewise isometry. Therefore the S[k] provide a fundamental link between the geometry of W and the dynamics of τ. All the dynamics of τ can be derived from S[k] dynamics and no step-k orbit needs to be considered beyond k = <N/2> at D. See Examples 4.2 & 4.3.

Definition 4.1 (The outer-billiards map τ) Suppose that P is a convex polygon in Euclidean space with origin internal to P. If p is a point external to P that does not lie on a blue 'trailing edge' of P, then the (clockwise) outer-billiards image of p is $\tau(p) = 2c-p$ where c is the nearest clockwise vertex of P.



Since τ is a reflection, the period of a regular N-gon P must be even -so all points inside P will have this same even period q. But if N is even, the center could have (prime) period q/2 because P can map to itself inverted after p/2 iterations. When N is odd its symmetry group $\mathcal{D}_N$ is based on vertices and sides instead of pairs of vertices, so all points would have the same period q.

Example 4.1 (S[2] for N = 7). By Lemma 4.1 to follow, cS[2] will have an orbit that advances two vertices on each iteration - so it will be period 7. The period of S[2] itself must be even so all other points will be period 14 as shown in the right. After 7 iterations S[2] will be inverted so the original offset from the center will be negated by the second round of 7 iterations.



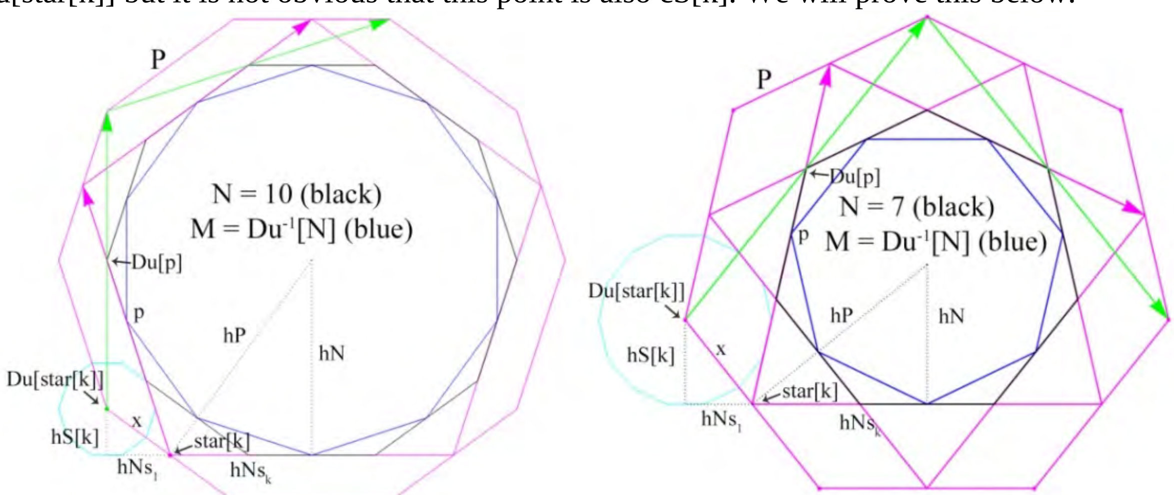
Note that $\tau^k(p)$ will always have the form $2Q + (-1)^k p$ where Q is a sum of k vertices of N with alternating sign. So the center point is period 7 iff $\tau^7(cS[2]) = 2Q - cS[2] = cS[2]$. This has only one solution which is Q = cS[2]. For all points in S[2] (including cS[2]), any 'second round' of 7 iterations will have the same Q (except for sign), so the concatenation yields $Q' = 0$ and $\tau^{14}(p) = p$. This is a 'stable' period 14 orbit for all points except cS[2].

Based on our remarks earlier this 'period doubling' of a regular N-gon can only occur when N is even and it provides an easy way to find the center of such polygons.

Lemma 4.1 (Canonical Orbits of S[k] centers) For any regular N-gon, cS[k] has an outer-billiards orbit that advances by k vertices on each iteration, so it has period N/(k,N).

Proof: We will show that there is a height/radius ‘duality’ between cS[k] and star[k] defined by **Du[x] = RotationTransform[-Pi/N, {0,0}][x*rN/hN]**. For any N-gon P centered at the origin, Du[P] will map midpoints of edges of P to the adjacent (cw) vertex of P. Therefore Du can be used to map an edge-based orbit of any star[k] point to a vertex-based orbit of Du[star[k]] as shown here in magenta and green for N = 10 and N = 7. (The green orbits are shortened here.)

Every star[k] point is a vertex of an {N,k} star polygon embedded in {N,<N/2>}. Therefore this star polygon defines an ‘edge-orbit’ O_k which coincides with the edges of N. This orbit advances k edges on each iteration so it has period N/(k,N). (Since k = 2 here the periods are 5 and 7.) Because{N,k} inherits the rotational symmery of N, this magenta orbit will extend equal distance on either side of the midpoint of each edge, so O_k also defines an outer-billiards orbit relative to M -which has vertices equal to the midpoints of N. Since Du[M] = N, it follows that Du[O_k] will be an outer-billiards orbit around N with period N/(k,N) as shown in green. Every vertex point in this orbit is Du[p] for some midpoint p in O_k. The initial point of these green orbits must be Du[star[k]] but it is not obvious that this point is also cS[k]. We will prove this below.



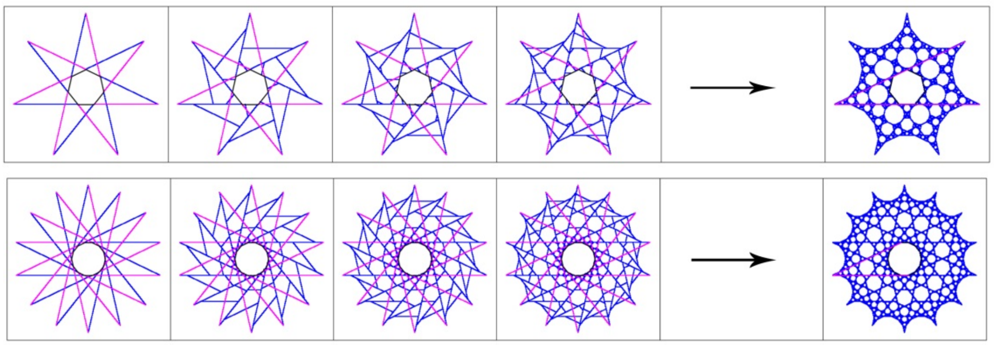
The polygons P were not needed earlier, but they will be useful here. P is defined to be the unique N-gon centered at the origin with star[k] as its midpoint. Therefore Du[star[k]] is a vertex of P and we will show that it must also be cS[k]. The length x shown here is sP/2 which is hPs_1(where s_1 is with respect to either N or P). As in the First Family Theorem, the displacement of cS[k] should be sN/2 = hNs_1. Therefore $hS[k]^2 = (hPs_1)^2 - (hNs_1)^2$, so $hS[k] = s_1\sqrt{hP^2 - hN^2}$.

But $hP^2 = hN^2(1+s_k^2)$ so $hS[k] = s_1\sqrt{hN^2(1+s_k^2) - hN^2} = s_1 hNs_k$. When N is even this is hS[k] but when is odd S[k] is a 2N-gon and hS[k] = hNs_2s_{2k} of 2N. But s_1 of N is s_2 of 2N and s_k of N is s_{2k} of 2N, so the same calculations yield hS[k] = hNs_2s_{2k} of 2N as in the First Family Theorem. Therefore cS[k] is always Du[star[k]] and the period of the orbit of cS[k] will be N/(k,N). □

Every orbit around N has a matching step sequence and the study of these sequences is part of symbolic dynamics. The calculations in the Appendix make use of these sequences. One of the most important issues in any dynamical system is to find connections between dynamical (temporal) scaling and geometric scaling. This lemma is a first step in that direction,

Definition 4.2 (The outer-billiards singularity set W of a convex polygon P)
Let $W_0 = \bigcup E_j$ where the E_j are the (open) extended edges of P. The level-k (forward) web is $W_k = \bigcup_{j=0}^{k} \tau^{-j}(W_0)$ and the level-k (inverse) web is $W_k^i = \bigcup_{j=0}^{k} \tau^{j}(W_0)$. $W = \lim_{k\to\infty} W_k$ and $W^i = \lim_{k\to\infty} W_k^i$
For a regular N-gon, τ^{-1} is τ applied to a horizontal reflection of N, so W_k and W_k^i are also related by a simple reflection and it is our convention to first generate W_k^i by mapping the 'forward' extended edges under τ and if desired a reflection gives W_k also. In the limit W and W^i must be identical but at every iteration they differ, so it is efficient to utilize both for analysis.

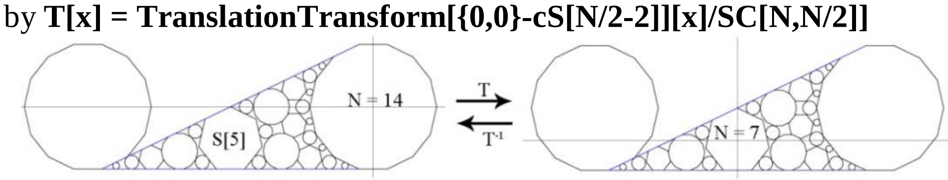
Example 4.2 (The star polygon webs of N = 7 and N = 14) .
The forward and trailing edges are shown in blue and magenta. Here we generate the level-k (inverse) webs W_k^i by iterating the blue forward edges under τ^k for k = 0,1,2,3 and 50. The magenta trailing edges are shown for reference. At every stage these images could be enhanced by taking the union with the horizontal reflection. In the limit it would not matter.



We call these 'generalized star polygons'. They retain the dihedral symmetry group $\mathcal{D}_N$ of N. By symmetry our 'region of interest' can be restricted to the regions outlined on the right – which always run from D to matching D. It is easy to show that these 'inner-star' regions are invariant under τ. In fact their step-sequences cannot exceed that of D – which is step-<N/2> by Lemma 4.1. In [VS], the authors give evidence for the fact that these regions bounded by D tiles, can serve as a 'template' for the global dynamics.

When N is twice-odd, the geometry of these two default regions should be equivalent because the equivalence of cyclotomic fields implies that the M tile of N can be regarded as N/2 under a scaling and change of origin. The Scaling Lemma gives the equivalence of scales under scale[k] of N/2 = scale[2k]/scale[2] of N = scale[2k]/SC(N,N/2) = scale[2k]·GenScale[N/2]/GenScale[N].

Lemma 4.2 (Twice-odd Lemma) For N twice-odd the First Families and webs of N and N/2 are related by **T[x] = TranslationTransform[{0,0}-cS[N/2-2]][x]/SC[N,N/2]]**



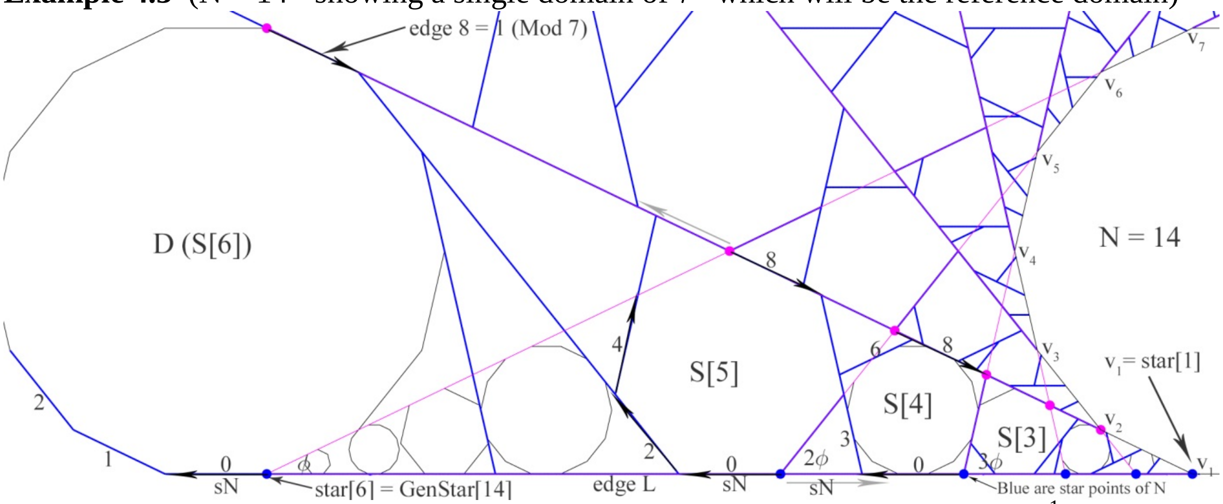
Proof: T maps M = S[N/2-2] of N to the origin and then scales it to be a gender change of N and Lemma 2.3 shows that this is consistent with the First Family Theorem since M is an odd S[k]. Because T and T^{-1} are affine transformations they will commute with the affine transformation τ and hence preserve the web W. □

Evolution of the Web

For practical and theoretical reasons it is important to understand how the extended edges of N evolve under τ.The problem is that there are N-domains which map to each other and we want to describe how a single domain evolves. In the graphical iteration of a function of one variable the y = x line is used to swap domain and range. This is called 'cobwebbing'. For rational piecewise isometries such as τ this cobwebbing typically involves a translation or 'shear' followed by a rotation to swap domain and range and prepare for the next iteration. This is also a recipe for constructing N and the two mappings of the Appendix use a shear and rotation to mimic W.

We will show that W can be generated by an iteration of 'shear and rotate' where the shear is of constant magnitude sN and the rotation angle is variable of the form $k'\phi$ where $\phi = 2\pi/N$ and $k' = N/2-k$ is the 'local index' of star[k] - so $k'\phi$ is the 'star-angle' determined by star[k].

Example 4.3 (N = 14 - showing a single domain of τ^{-1} which will be the reference domain)



By symmetry it is sufficient to track the evolution of one of the domains of τ or τ^{-1}. Our canonical choice is the (open) domain of τ^{-1} shown here between two blue extended edges of N. This is called Domv$_1$ because for all p, $\tau^{-1}(p) = 2v_1 - p$ where v_1= star[1] of N. This domain will intersect <N/2> magenta domains of τ defined by the trailing edges of N and the star[k] points. Our goal is to track the evolution of these star[k] domains under τ. As p approaches the horizontal edge L from the top, $\tau^{-1}(p) = 2v_1 - p$ becomes a horizontal shear of magnitude $2v_1$ = sN, so under τ, points in Domv$_1$ will feel an outward shear of this magnitude as they approach edge L - and points below edge L will feel the opposite shear - as shown by the arrows above.

The blue star points on L have no image under τ or τ^{-1} but by continuity they must feel this same shear. Therefore each star[k] and shear can be used to define a 'level-0 base' relative to S[k] as shown here. Note that the S[k] centers of the First Family Theorem are consistent with these shears. Recall that the local index k' of S[k] was defined so that star[k] of N is star[k'] of S[k]. This means that the proposed rotation angle $k'\phi$ is the angle between edge L and the next step-k' edge of S[k]. Therefore rotating the level-0 base edge of S[k] by $k'\phi$ will align a segment of edge L with an edge of S[k]. Under τ, this segment will feel the sN shear so this can be repeated recursively - but the resulting web evolution must be combined with the matching iteration of the top edge. In this sense D acts as a reflected copy of N and defines a competing web.

At GenStar[N], the local index is $k' = N/2-\langle N/2\rangle = 1$ so the rotation angle will be $\phi = 2\pi/N$ to match that of N. This is the rotation that will allow the hypothetical edge 1 shown above to be a horizontal edge and hence experience the shear sN. The top and bottom shears are ϕ apart because they are on consecutive edges. Therefore for D, the top shear will be relative to edge 8 (1 mod N/2). The matching S[k] edges will have this same 8-step orientation with respect to the level-0 horizontal base. This value will always be even when N is twice-odd, so the primary and secondary cycles are synchronized and the S[k] are formed in a redundant step-k' fashion.

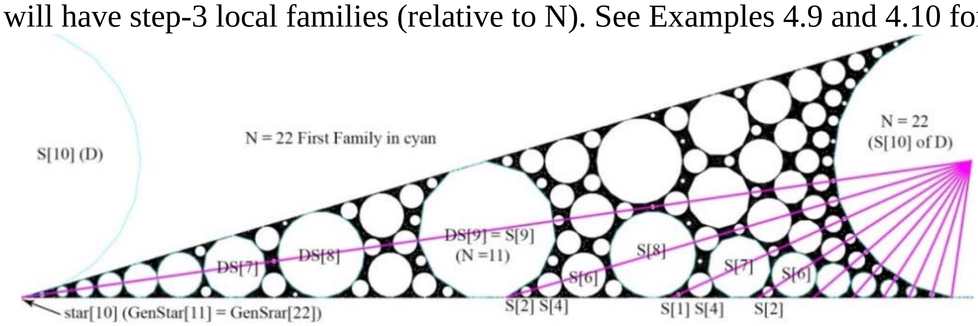
These web cycles will have period $N/\gcd(k',N)$ so when $\gcd(k',N) > 1$, the cycles will be shortened. This occurs here for N = 14, because the odd S[k] have $k' = N/2-k$ even - so the rotational periods will be N/2 for both the primary and secondary cycles. This means that S[1], S[3] and S[5] will be heptagons. These can be regarded as 'mutations' relative to the even S[k] where $\gcd(k',N) = 1$ because k' is odd. All twice-odd N-gons will have this 'androgynous' nature.

When N is twice-even the star angle and shears are unchanged but now it takes an odd number of steps to go from the bottom edge to the top shear, so the even and odd S[k] are typically N-gons. Here the mutation condition is $\gcd(k',N) > 2$ because if $\gcd(k',N) = 2$, the two N/2 periods will not be synchronized – so S[4] of N = 12 with $k' = 2$ will not be mutated.

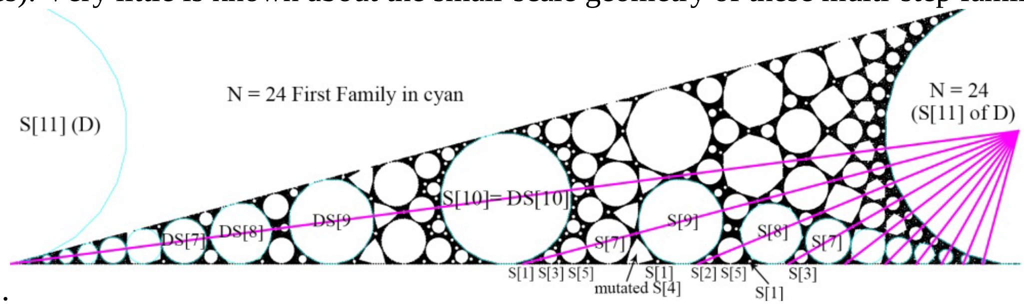
When N is odd the (relative) shears are unchanged from the even case and the star angles are compatible since they are of the form $(\phi/2)(N-2k) = \phi(N/2-k)$. So the S[k] local indices are $k' = N-2k$ and D = S[<N/2>] will have index 1 and rotation angle $\phi/2$. Therefore D will be a 2N-gon with the same side as N. The S[k] will also be 2N-gons so mutations will occur iff $\gcd(N-2k,2N) > 1$ so S[6] of N = 15, will have $k' = 3$ and be mutated since $\gcd(3,30) = 3$.

In all cases the web rotational periods may be reduced depending on $\gcd(k'N)$ and any such reduction may yield a 'mutated' version of S[k] as described in the Mutation Conjecture to follow. (These web cycles are useful concepts arising from symmetry - but they are not τ-orbits. For piecewise mappings, standard theories of evolution for stable and unstable manifolds must be modified to describe local web evolution. This has not been done yet so there is no theory that describes the evolution of τ step sequences of an arbitrary N-gon – beyond the elementary results of Lemma 4.1.The only non-trivial cases known are N = 5 and N = 8 in [T],[BC] and [S].)

Example 4.4 (The web of N = 22) Since all S[k] are formed in a step-k' fashion in the web they will have local webs which are also step-k'. This is consistent with the Twice-odd Lemma which says that the local family of S[9] will be scaled copies of the S[k] of N = 11 – which are S[2k] of N. The magenta lines below track the resulting local families. For example S[8] has $k' = N/2-k = 3$ so it will have step-3 local families (relative to N). See Examples 4.9 and 4.10 for details.



Example 4.5 (The web of N = 24) Since N is twice-even all the ideal S[k] are N-gons but mutations will occur where gcd(N/2-k, N) > 2 so M = S[10] is spared from mutation along with S[2], but all other S[k] are mutated except S[1],S[5], S[7] and S[11]. Even though M looks 'normal' it was formed in a step-2 fashion, so its local family is step-2 relative to N. S[9] has the same index 3 as S[8] above, but it is mutated since gcd(3,24) = 3. These mutations are always consistent with the step-k′ evolution, so the mutated S[9] will have step-3 local families (on both sides). Very little is known about the small-scale geometry of these multi-step families.



The digital-filter map discussed in the Appendix will work for multi-step rotations, and we have been able to verify that using the maximal digital-filter step-5 (24/4 -1) for N = 24 will reproduce the local geometry of S[10], but almost nothing is known about the general case. It does appear that outer-billiards is just a step-1 version of a much broader class of mappings. For N = 60 there are 14 non-trivial step-k mappings (60/4 -1) and the maximal 14-step web appears to be identical to the local geometry of the S[28] M-tile of N = 60. See Appendix F of [H3].

Conjecture for period-based mutations of S[k]

(i) When N is twice-odd, S[k] will have local index k′ = N/2-k so the rotational period of S[k] in the web will be N/gcd(k′,N) and S[k] will be mutated iff this period is less than N.

(ii) When N is twice-even the local index of S[k] will be identical to k′ above so the rotational period is also identical, but now it takes an odd number of rotational steps to reach the 'retrograde' top edge so the rotational period can be N/2 with no mutation of S[k]. Therefore S[k] is mutated iff N/gcd(k′,N) > 2.

(iii) When N is odd, S[k] will have local index k′ = N-2k which is twice the index above because the S[k] are now 2N-gons. The rotational period of the S[k] local web will be 2N/gcd(k′,2N) and S[k] will be mutated iff this period is less than 2N.

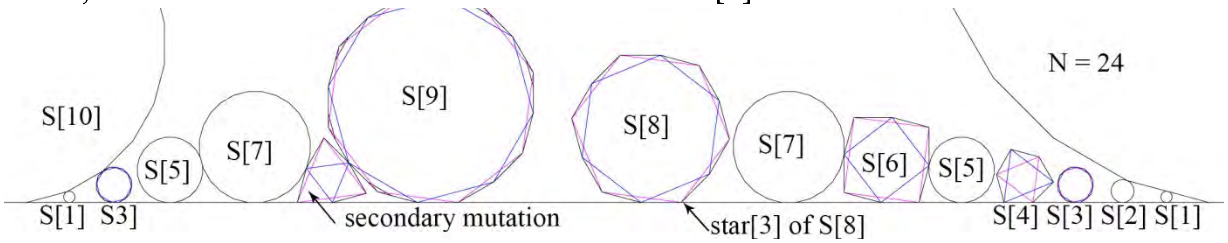
(iv) In all cases the mutated S[k] will be an equilateral 2m-gon which consists of the interleave of two regular m-gons where m is the (reduced) rotational period. These two m-gons will have (unequal) radii consisting of the radial distance from two star points of S[k] to cS[k].

The S[k] mutations of N = 24 are shown in the table below. From Lemma 4.1 the τ periods of cS[k] are N/(k,N) and these centers are unchanged by mutations. Here we compare these periods with the web rotational periods. In most cases a shortened τ-period predicts a mutation but that fails for S[10] and S[2] where gcd(N/2-k,N) = 2.

Table 4.1 Outer- billiards periods and web rotational periods for the S[k] of N = 24

Tile (S[k)	S[11]	S[10]	S[9]	S[8]	S[7]	S[6]	S[5]	S[4]	S[3]	S[2]	S[1]
Period N/(k,N)	24	12	8	3	24	4	24	6	8	12	24
Period N/(N/2-k,N)	24	12	8	6	24	4	24	3	8	12	24
Mutated ?	N	N	Y	Y	N	Y	N	Y	Y	N	N

Example 4.6 (Mutations of S[k] for N = 24) Since S[k] mutations are 'unfinished' local webs, they will be consistent with the underlying regular S[k]. The star points of S[k] are preserved under rational rotations so the sides of the mutated version will still be the difference of two star points. For S[8] below the two star points are right-side star[3] and the left-side star[1]. Since S[8] has local index k′ = N/2-8 = 4 , star[4] of S[8] will be star[8] of N. This point is not shown below, but it is the reference for the 'level-0 base' for S[8].



The First Family Theorem says that the midpoint of S[k] will be displaced by sN/2 from star[8] of N. This will remain true for any mutation because the sN shear is unchanged. The problem is that the rotational web period for S[8] is now 24/4 = 6, so S[8] will be based on two regular hexagons as described in (iv) of the Mutation Conjecture. These hexagons are defined by the bottom and top shears so each will be step-4 but not synchronized since N is twice-even. Here the radii are defined by star[3] and star[1] relative to cS[8] – and these star points define the side length. (We call the result a 'semi-regular' dodecagon in [H3] because it is equilateral and has dihedral symmetry group $\mathcal{D}_6$ rather than $\mathcal{D}_{12}$. At the origin it seems to have bounded dynamics.)

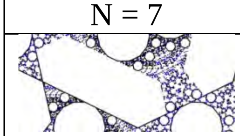
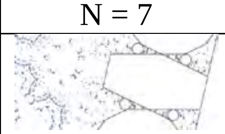
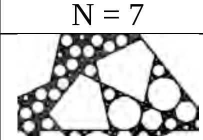
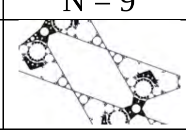
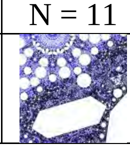
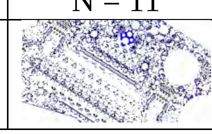
Here are the steps to construct this mutated S[8] based on the known height and center of S[8].
(i) Find the star[3] point: MidS8= {cS[8][[1]],-hN}; star[3] = MidS8+{Tan[3·Pi/24]·hS[8],0};
(ii) H1=RotateVertex[star[3], 6, cS[8]] (magenta); H2 =RotateVertex[star[1],6, cS[8]] (blue)
(iii) MuS8 = Riffle[H1,H2] (black) (This weaves them as in a card shuffle.)

As indicated above these mutations are incomplete local webs which can be regarded as 'scaffolding' for the underlying S[k] tiles. The ideal S[k] will always be in the convex envelope of this scaffolding so in this sense the S[k] are preserved under W.

The general web

Because of the multi-step origin of the S[k] the small-scale web may be very different from the orderly First Family structure. The limiting 'tiles' could be points - which must have non-periodic orbits, or possibly lines similar to the structures that appear for N = 11 on the right below. No one has ruled out the possibility of limiting regions with non-zero Lebesgue measure. This is a long-standing open question in the phase-space geometry of Hamiltonian systems. Any non-limiting tile must be convex with edges parallel to those of N, so it is easy to see that the D tiles are maximal among regular polygons – and in fact rings of these tiles must exist at all radial distances so the dynamics in any finite region must be bounded.

Example 4.7 Some non-regular tiles in W_k (click to enlarge)

N = 7	N = 7	N = 7	N = 9	N = 11	N = 11
					

Local webs of the S[k] – in-situ vs. in-vitro evolution

Every polygon P can be regarded as an ‘in-vitro parent’ at the origin, or P could be just a tile in the web or family of another polygon and we want to know how it evolves ‘in-situ’. When N is twice-odd, the transformation T of Lemma 4.2 relates the in-situ M tile of N with its in-vitro form as N/2. When N is even the D tile is a retrograde version of N so the in-situ to in-vitro transformation T is a reflection about cM. For most P the two viewpoints are very different.

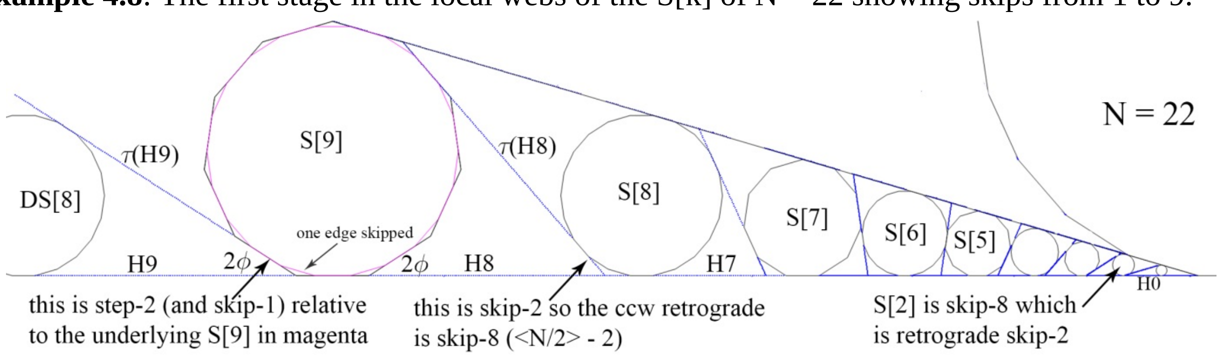
For linear or quadratic N-gons, the evolution of the S[k] is tied to the evolution of N because W is either linear or self-similar. The 4k+1 Conjecture to follow states that when N is 4k+1 (or 8k+2), there will always be families of D[k] or M[k] converging to GenStar, but the conjecture says nothing about the evolution of any S[k] except S[1] and S[2].

The previous analysis makes it clear that there are no ‘normal’ S[k] when it comes to web evolution because these S[k] are generated in the web in a multi-step fashion (except for D). This origin will have a lasting effect on the local geometry of S[k]. Very little is known about ‘multi-step’ geometry since τ and the web are basically single-step processes. Here we will contrast the ‘normal’ S[1], S[2] evolution for N-even, with the case of N-odd where S[1] and S[2] are a step-2 ‘out-of context’ family since they are congruent to DS[2] and DS[4].

We will once again consider the web W to be the invariant ‘star-polygon’ web which is bounded by rings of D tiles. W can be generated by iterating the forward edges of N under a ‘normal’ clockwise τ - but when N is even, D is a reflection of N, so from D’s perspective W is a ‘right-side’ web which evolves counter-clockwise. This is just a viewpoint distinction because W is the same in both cases, but if we apply this left-right distinction to an S[k] tile, the two local webs will typically be different and it may be useful to consider the ‘normal’ left-side local web (as if it were N) as well as the right-side web from the ccw D perspective. In terms of web evolution they are both iterated the same way under τ, but now they are different intervals.

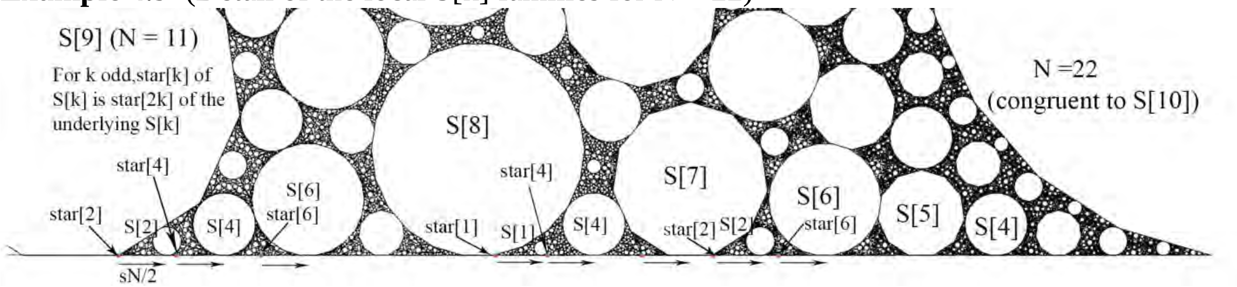
As the local indices of the S[k] grow in size it may be useful to use ‘retrograde’ ccw webs because their indices are the complement of the cw indices. For N even the S[k] local indices are N/2-k and this describes the rotational steps between S[k] and star[k]. This transition skips N/2-k-1 = <N/2> - k edges and we will use these ‘edge skips’ here instead of ‘rotational steps’.

Example 4.8: The first stage in the local webs of the S[k] of N = 22 showing skips from 1 to 9.



Since clockwise and ccw rotations have edge skips differing by <N/2>, the ccw web of each S[k] will skip-k edges - to match its orbital rotation around N. This will yield valuable information about the geometry of W local to N, because S[1] and S[2] have more manageable ccw webs. If the intervals H_k are taken to be disjoint and their union is one complete extended edge, then the local webs will also be disjoint and W will be the disjoint union of the local webs.

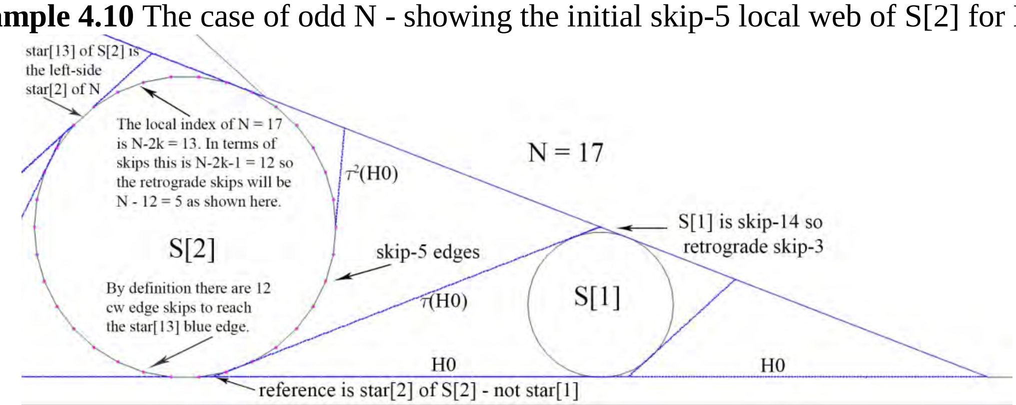
Example 4.9 (Detail of the local S[k] families for N = 22)



It is easy to construct these multi-step families because the relevant (magenta) star points are known. Since each of these (parent) S[k] can construct N in a symmetric fashion as in the First Family Theorem, the displacement of the local S[k] from star[k] is always sN/2 as shown here. For consistency gender-change mutations are ignored by labeling star points with respect to the underlying S[k] - so star[1] of S[9] and S[7] are labeled as star[2]. In terms of local family evolution S[9] and S[7] are step-2 and step-4, so for S[9] the first two relevant star points are star[2] and star[4] – while for S[7] the first two relevant star points are star[2] and star[6]. In both cases these star points define a local S[2], so these two versions of S[2] will have very different dynamics. Click above to see the contrast. This is why so little is known about the local geometry of these families. At least for S[9] it is possible to evoke the Twice-odd Lemma and work with N = 11 – where it is possible to conclude (see below) that the retrograde skips are 2k+1, so S[2] of N = 11 (which is labeled S[4] above) is (left-side) skip-5.

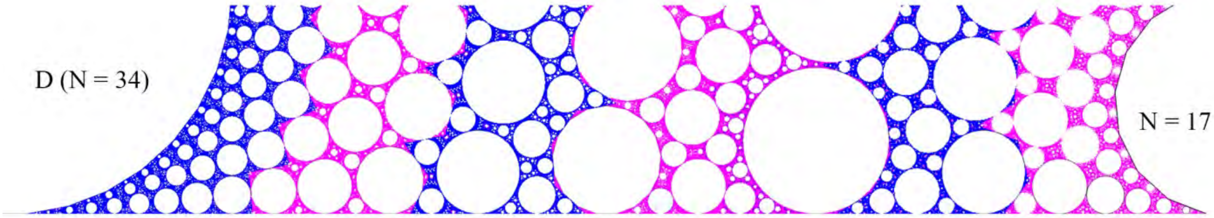
When N is odd the index of S[k] is N-2k steps which is N-2k-1 skips. Since the S[k] are 2N-gons the retrograde skips are relative to N, so they are N-(N-2k-1) = 2k + 1. Therefore the skips are more than doubled from the N-even case. This increase is due to the fact that the S[k] are 2N-gons - and this is the due to the fact that N itself is mutated and will have a step-2 local family.

Example 4.10 The case of odd N - showing the initial skip-5 local web of S[2] for N = 17



This 5-edge skip for S[2] will be universal for N odd just like the 2-skip is universal for N even. For small k, these right-side ccw S[k] webs will be more manageable that the traditional left side cw webs which have higher skip rates. For S[2] above, the left-side cw web will be skip-12.

Example 4.11 The large-scale geometry of N = 17 showing invariant regions. (In 1796 Carl Gauss proved that N =17 can be constructed with compass and straight-edge since $17 = 2^{2^2} + 1$.)



For N-odd, the First Family includes even and odd cases so it contains two very different edge geometries at D and at N. At D, S[2] is known as DS[2] and by the results above it will have a fairly well-behaved skip-2 local web that will include DS[1] – which has the potential to be a 2^{nd} generation N = 17. The 4k+1 conjecture below predicts that for N = 17, this process can be continued with generations of D[k] and M[k] tiles converging to GenStar[17]. The edge geometry of N =17 itself is also dominated by the local web of S[2] which contains S[1], but these are 2N-gons and S[2] has a skip-5 geometry that is not so well-behaved.

The 4k+1 Conjecture (part (i) is not conjecture)

(i) For any regular N-gon with N > 4 Definition 3.3 describes a well-defined (ideal) sequence of M[k] & D[k] tiles converging to GenStar[N] with M[1] = DS[1] and D[1] = DS[2] and for any positive integer k, hM[k+1]/hM[k] = hD[k+1]/hD[k] = GenScale[N]

(ii) When N = 4k+1 we conjecture that the M[k] and D[k] in part (i) exist under the outer-billiards map τ.

(iii) When N = 4k+1 we also conjecture that the ratio of the τ-periods of cM[k] , cM[k-1] and cD[k], cD[k-1] approaches N + 1.

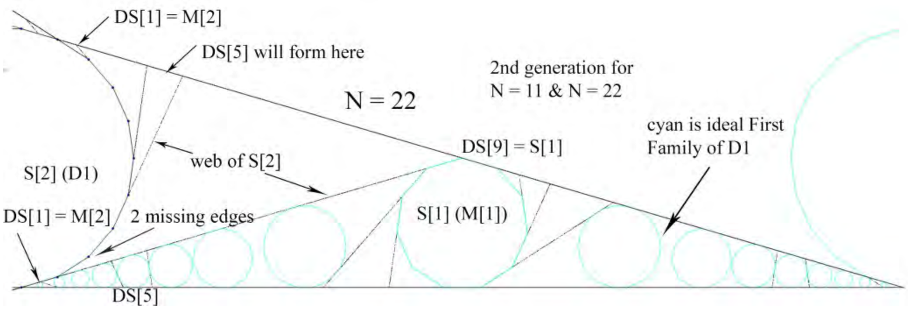
It appears that this conjecture is the recursive form of a more fundamental conjecture about the local web evolution of the S[2] tiles for an arbitrary N-gon.

Definition 4.3 (The DS[k] of S[2]) For a regular N-gon, the tiles on the edges of N which exist in the limiting web W and are strongly conforming to star[1] of S[2] for N-even or star[2] for N-odd, will be called the 'DS[k] tiles of S[2]' or simply DS[k] when S[2] is understood. When N is even the DS[k] conforming to star[1] of S[2] will be identical to the S[k] of the First Family of S[2]. When N is odd the DS[k] strongly conforming to star[2] will be as described below.

Edge Conjecture. For an arbitrary regular N-gon, every potential DS[k] tile on the edges of N which satisfies the Rule of 4 for N-even or the Rule of 8 for N-odd, will be among those that exist in the web W. The Rule of 4 says that counting down from S[1] at DS[N/2-2], DS[k] will at least exist mod 4 and The Rule of 8 says that counting down from S[1] at DS[N-4], DS[k] will exist at least mod 8. (In this case the DS[k] must conform to star[2] of S[2] so they will be 2N-gons with hDS[k]/hS[k] = 1/scale[2] of 2N.Since the S[k] will have odd k values, they will be N-gons and this height relationship says that they will be congruent to the M tile of DS[k].)

These two rules are a direct consequence of the skip-2 and skip-5 local webs of S[2]. We begin with the N even case. As noted above the local web of S[2] will skip <N/2>-2 edges, so its right-side web will skip 2 edges – to match its orbit around N. Therefore any existing extended edge will be followed by 2 blank edges as shown here for N = 22 and this implies the Rule of 4 for the DS[k]. The countdown starts with S[1] at DS[9], so the guaranteed survivors will be DS[5] and DS[1]. This explains why an isolated DS[1] and DS[5] survive in the 2nd generation of N = 11 in Example 1.2. This missing DS[2] and DS[3] imply that there will be no canonical 3rd generation.

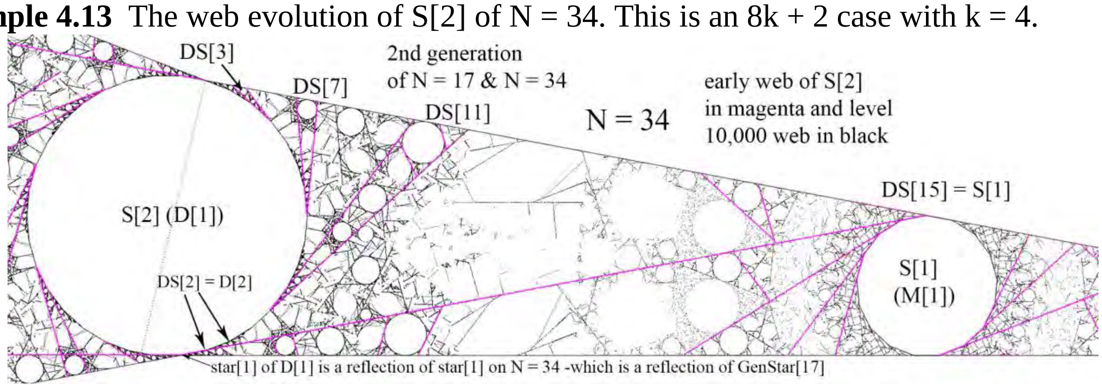
Example 4.12 The early web evolution of S[2] of N = 22.



Every regular N-gon for N even will have a similar Rule of 4 issue - with the likelihood of three missing DS[k] between existing DS[k]. N = 22 is typical of the 8k + 6 case where DS[1] will be among the survivors. Since S[1] is always a survivor at DS[N/2-2] it is easy to predict which DS[k] will survive. N = 30 will be similar to N= 22. The 8k+ 2 (4k+1) case is the only one where DS[3] is a guaranteed survivor and it is easy to prove that for N twice-odd, every odd S[k] will have DS[2] in its First Family and the even S[k]s will have S[1] in their First Family, so DS[3] can account for DS[2]s and these in turn can generate DS[1]s as part of their webs.

The Edge Conjecture can be applied recursively with S[2] as the new N so D[2] is the new S[2]. This recursion would be feasible if D[2] has a local web evolution similar to D[1] and there is evidence that the skip-2 web evolution of the D[k] is typically inherited. For the 8k+2 case, this would imply part (ii) of the 4k+1 conjecture and we will show here that rotational symmetry and the Rule of 4 would imply the D[k] & M[k] N +1 temporal scaling of part (iii).

Example 4.13 The web evolution of S[2] of N = 34. This is an 8k + 2 case with k = 4.



Each DS[3] 'cluster' shown here contains 2 D[2]s, and there will always be k clusters on the right side of S[2], so the right-side count is 2k D[2]s. Relative to the blue line of symmetry, the back side will also have 2k D[2]s, with one cluster divided between the sides. This yields 4k+ 2

D[2]s which is the N/2 + 1 temporal scaling predicted by the 4k+ 1 conjecture. This predicted convergence at star[1] of S[2] is congruent to the ideal First Family convergence at GenStar from Definition 3.3, because the edges of N must have reflective symmetry and star[1] of N is a reflection of GenStar at D. Therefore this N/2+1 temporal scaling can be regarded as a N/2 scaling for D[k] itself along with the '+1' which is the outlier D[k+1] at GenStar (or star[1] of D[k]). This D[k+1] becomes the foundation for the next generation.

Table 4.1 Rule of 4 - surviving 2nd generation tiles on the edge of N for N even (To apply the Rule of 4, subtract 4 from previous DS[j]. All existing DS[k] will be strongly conforming.)

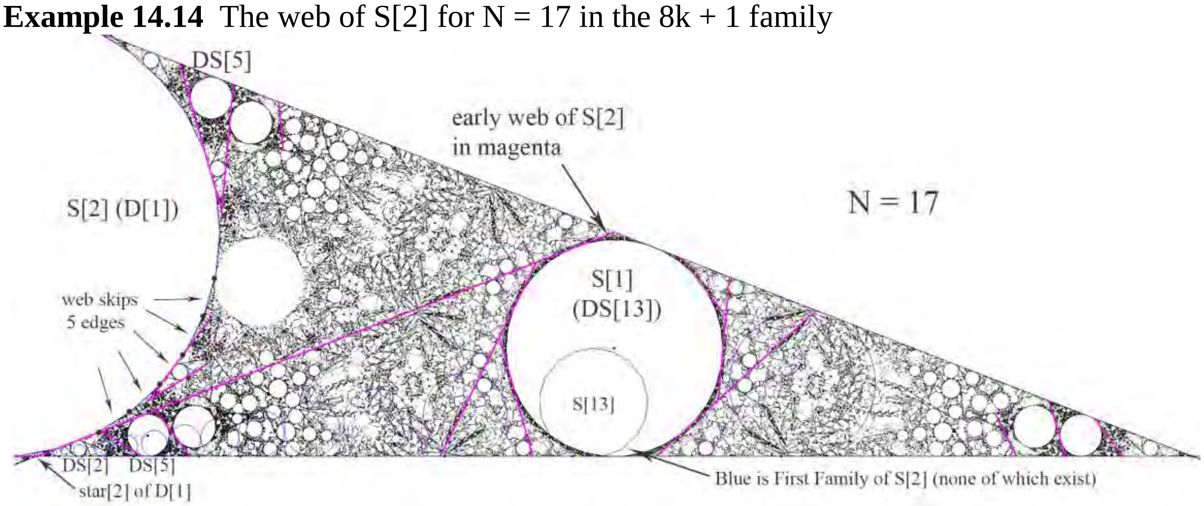
N	DS[N/2-2] (S[1])?	Rule of 4	DS[3] ?	DS[2] (D2) ?	DS[1] (M2) ?
8k (8, 24)	Y	…….	No if >8	Y	No if > 8
8k + 2 (10, 34)	Y	…….	Y →	Y →	Y
8k + 4 (12,20)	Y	…….	No	No if >12	No if >12
8k + 6 (14,22)	Y	…….	No	No if > 14	Y

The 'worst' case scenario for N even appears to be the 8k+4 case (with the exception of N = 12) where the chain of S[k] does not begin until DS[4], so N = 20 has no D[2] or M[2] tiles to form canonical generations on the edges of N. The extreme 8k + 6 and 8k cases are not much better because the former has isolated M[2]s and the latter has isolated D[2]s.

The web evolution of S[2] for N odd and the Rule of 8

All web evolution is based on the edges of N, so for the N even, the basic unit of rotation for S[2] is the same as N. This is no longer true when N is odd because S[2] and S[1] are now 2N-gons. As in Example 4.13, this means that the retrograde skips will be 2k+1 instead of k and the Rule of 4 will become the Rule of 8. Therefore the new DS[k] families will be sparse compared with the traditional S[k]. For N = 17 shown here S[1] is at DS[13] so the Rule of 8 predicts a DS[5]. This will always be true for the 8k+1 family. (It is common for members of this family to also upport a 'volunteer' DS[2] because for N twice-odd DS[5] will always have DS[2] in its First Family - so they have compatible web evolution.)

Example 14.14 The web of S[2] for N = 17 in the 8k + 1 family

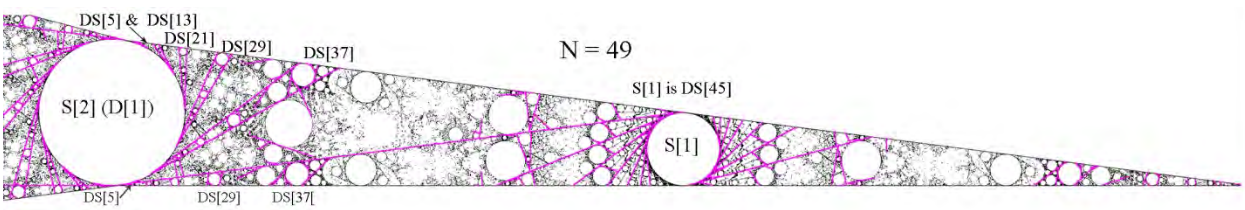


The scaling between the S[k] and DS[k] is determined by the fact that they both share star[k] – but the reference for S[k] is star[1] while the reference for DS[k] is star[2]. Therefore the height scaling is hDS[k]/hS[k] = 1/scale[2] of 2N (which is the same as 2+ GenScale[N]). By the Two-Star Lemma this will imply that the DS[k] are 2N-gons.

Because of the Rule of 8, all the predicted DS[k] will be odd (since S[1] is DS[N-4]). Therefore the matching S[k] will be N-gons, and the scale[2] relationship is the same as D-M scaling. This is illustrated above with the dotted lines which show that hDS[k] = hS[k] + rS[k]. The scaling for DS[2] and S[2] must also be scale[2], but now the genders match, so hDS[2] = hSx[2] + rSx[2] where Sx[2] is the gender dual of S[2] – with the same center and height as S[2].

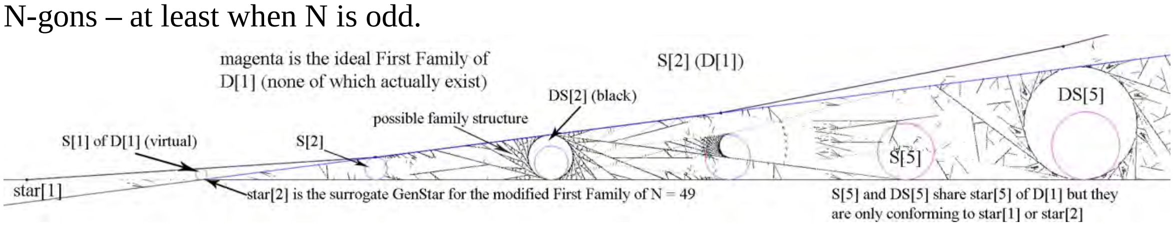
For N = 17 there is no overlap between the S[k] and DS[k] but when DS[1] exists (as with the 8k+5 family) it will match S[2] exactly since hS[2]/hS[1] = 1/scale[2] of D[1].

Example 4.15 -The web evolution of S[2] for N = 49 – which is also in the 8k+1 family.



N = 49 has algebraic complexity 21 which is 7 times the complexity of N = 7. In terms of edge geometry, N = 49 is in the 8k+1 family like N = 17 so the Rule of 8 predicts the existence of S[1] at DS[45] as well as DS[37], DS[29], DS[21], DS[13] and DS[5] as charter members of the modified First Family of S[2]. We call these 'scale[2]' families.

In the enlargement below it is clear that DS[2] exists but DS[1] does not exist. Since DS[2] shares the expected star[2] and star[4] of D[1], the Two-Star Lemma says that hDS[2] = (starD1[4][[1]]-starD1[2][[1]])/(Tan[47π/98 + Tan[4π/98]) ≈ 0.000265730947904.
This strong star[2] conformity guarantees that hDS[2]/hS[2] will be 1/scale[2] of N = 98. As indicated earlier this DS[2] = D[2] tile is common for the 8k+1 family. Its local web typically has hints of extended structure as illustrated below. In terms of gender, DS[2] is somewhat androgynous because it is an edge tile of D[1] and typically the edge -based tiles of a 2N-gon are N-gons – at least when N is odd.



The First Family Scaling Lemma says that hS[1]/hS[2] of N = 49 will be scale[2] of N – not scale[2] of 2N, so S[1] and S[2] above do not have the same height relationship as S[1] and S[2] of D[1] and there is no 'nice' self-similarity. When N is odd, scale[2] is primitive so the 'ideal' S[1] and S[2] families are fundamentally different - yet S[1] and S[2] shown here must share the same local web on the edges of N so their families are connected in a non-trival fashion.

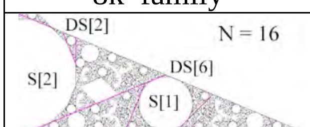
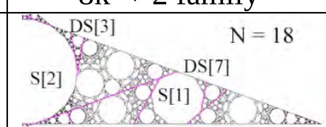
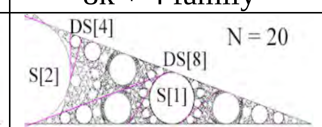
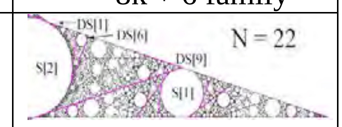
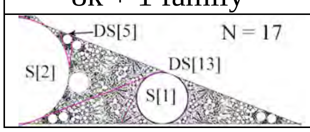
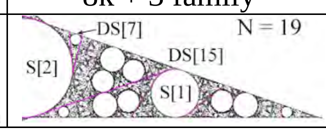
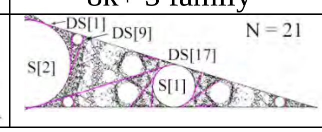
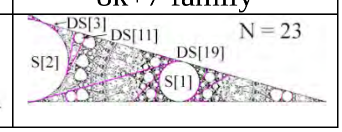
Table 4.2 Rule of 8 - surviving 2nd generation tiles on the edge of N for N odd (To apply the Rule of 8, subtract 8 from previous DS[j]. All existing DS[k] will be strongly conforming to star[2] of S[2].)

N	DS[N-4] (S[1]) ?	Rule of 8	DS[3] ?	DS[2]) ?	DS[1] ?
8k + 1 (9, 17, 25)	Y	…….	No	Y(?)	No
8k + 3 (11, 19,27)	Y	…….	No	No	No
8k + 5 (13,21,29)	Y	…….	No	No	Y
8k + 7 (15,23,31)	Y	…….	Y	No(?)	Y

To summarize, the Edge Conjecture can be used to predict strongly conforming DS[k] tiles on the edges of N and when N is even the DS[k] are a subset of the known First Family of S[2], but when N is odd the DS[k] are modified members of the First Family of S[2].Therefore it is now possible to make predictions about the geometry and dynamics on the edges of any regular N-gon and this may provide some insight into the overall evolution of W.

Below are examples of the 8 possible edge cases based on the Rule of 4 for N even and the Rule of 8 for N odd. It is always true that S[2] and S[1] are part of the First Family of N and it is no surprise that the local web of S[2] can generate the S[1] tile – but in the N-odd case, the existing DS[k] tiles have a scale[2] D-M relation with the odd S[k] of the First Family of S[2] – so the Rule of 8 seems to imply dynamics that are very different than the N-even case.

Table 4.3 A classification of web geometry on the edges of a regular N-gon - based on the Rule of 4 for N even (top) and the Rule of 8 for N odd (bottom). (This is the '8-fold way' for N-gons.)

8k family	8k + 2 family	8k + 4 family	8k + 6 family
			
8k + 1 family	**8k + 3 family**	**8k+ 5 family**	**8k+7 family**
			

The 4k+1 cases are ¼ of the N-gons and the corresponding mod-8 odd classes are 8k+1 and 8k+5 so again they are ¼ of the total. The matching twice-odds collapse down to just 8k+2 but this is still ¼ of the even mod-8 class. The 4k+1 conjecture and the Edge Conjecture say that these 8k + 2 N-gons should have a well-defined generation structure. Therefore 1/8 of all N-gons appear to have an edge geometry which is driven by sequences of self-similar D[k]-M[k] tiles with known geometric and temporal scaling.

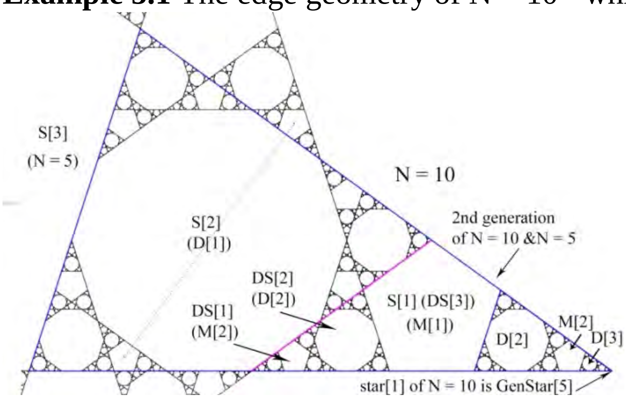
The long-standing 4k+1 conjecture should really be called the 8k+2 edge conjecture. It is not clear whether the matching 8k+1 or 8k+5 classes also have predictable local geometry. We will address this issue in [H9] and the examples to follow. Apparently members of the 8k+1 family have 'volunteer' DS[2] tiles which may have some form of extended family structure. In the 8k family DS[2] also exists and for N = 16 (Example 5.8) there appears to be an extended family structure.

Section 5 Examples of Singularity Sets

These examples will include the twice-odd pairs 5 & 10, 7 &14, 11 & 22 and 13 & 26 as well as the twice even cases of N = 12, 16 and 24.

N = 5 and N = 8 are the only non-trivial regular cases where the singularity sets have been studied in detail. In [T] (1995) S. Tabachnikov derived the fractal dimension of W for N = 5 using ‘normalization’ methods and symbolic dynamics and in [S2] (2006) R. Schwartz used similar methods for N = 8. In [BC] (2011) Bedaride and Cassaigne reproduced Tabachnikov’s results in the context of ‘language’ analysis and showed that N = 5 and N= 10 had equivalent sequences. In [H3] we give an independent analysis of the temporal scaling of N = 5 based on difference equations and this will be reproduced here in the context of N = 10.

Example 5.1 The edge geometry of N = 10 - which is the first member of the 8k+2 family

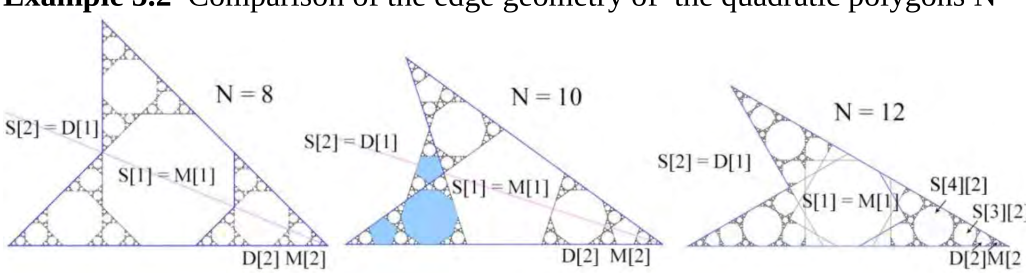


Generation	decagons - d_n	pentagons - p_n
1	1 (D[1])	1 (M[1])
2	5 = $3d_1 + 2p_1$	8 = $6d_1 + 2p_1$
3	31 = $3d_2 + 2p_2$	46 = $6d_2 + 2p_2$
4	185 = $3d_3 + 2p_3$	278 = $6d_3 + 2p_3$
n	$d_n = 3d_{n-1} + 2p_{n-1}$	$p_n = 6d_{n-1} + 2p_{n-1}$

Based on the Rule of Four, there will be k DS[3] ‘clusters’ on each side of S[2] and one cluster is divided by the line of symmetry to yield 4k+2 D[2]s for a growth of N/2+1. The 4k+1 conjecture predicts that subsequent D[k] will have a similar geometry so the limiting temporal scaling should be 6. The canonical GenStar[5] convergence of the D[k[and M[k] is shown above. This convergence appears to involve self-similar blue triangles which are anchored by D[k], so the geometric scaling of these triangles will be hD[k]/hD[k-1] = GenScale[5].

Under the assumption of self-similarity the difference equations in the table above are valid as shown in [H3]. These two difference equations relating decagons and pentagons can be combined together to yield a second-order equation $d_n = 5d_{n-1} + 6d_{n-2}$ which can be solved but it shows immediately that d_n / d_{n-1} must approach 6. This will be contrasted below with N = 8 and 12. Since the edges of any regular polygon have reflective symmetry relative to τ, there will always be a ‘3-dart’ configuration anchored by S[1] as shown here.

Example 5.2 Comparison of the edge geometry of the quadratic polygons N = 8, 10 and 12



These webs have just one non-trivial primitive geometric scale which is GenScale[N] (or GenScale[N/2] for N = 10). This is hD[k]/hD[k-1] = hM[k]/hM[k-1], so it is also the scale of the darts (or triangles) which are anchored by M[k] or D[k].

In each case the magenta 'renormalization' line shows how the initial dart (or triangle) is mapped to a self-similar version of itself under τ^k for some k. As expected the N = 8 and N = 12 cases are closely related since their cyclotomic fields are generated by $\{\sqrt{2},i\}$ and $\{\sqrt{3},i\}$.

(i) For N = 8 (&12), the D[k] and M[k] are identical except for size. There are 3 darts in this invariant region and they are anchored by D[k]s and each D[k] is surrounded by 3 M[k]s - so the M[k] have temporal scaling of 9.

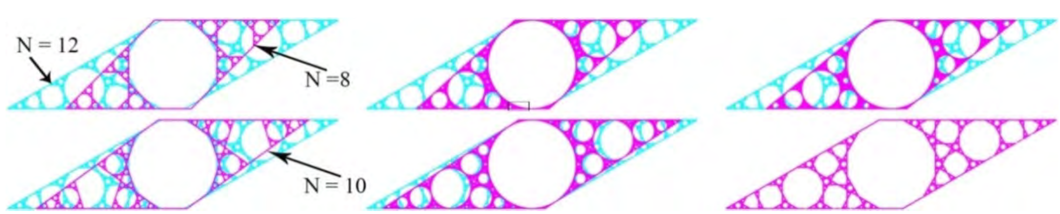
(ii) The temporal scaling of N = 10 is 6 as predicted by the 4k+1 conjecture and Example 5.1. Note that the 'next-generation' light-blue region shown here is composed of two overlapping 'towers' containing an M[k] and each M[k[is surrounded by 3 D[k+1]s for a temporal scaling of 6 for the D[k]. Since these towers form a sequence converging to star[1] of M[1] this helps to explain why the D[k]s overall should have this same scaling. This is a non-trivial fact and as explained earlier, the key issue is the relationship between the decagons and pentagons.

(iii) For N = 12, each dart is anchored by an S[4] and each S[4] is surrounded by 3 S[3]s for a combined scaling of 9, and in the limit each S[3] will account for 3 M[k]s, so the M[k] scale by 27. This case is also not trivial and it is covered in more detail in Example 5.6.

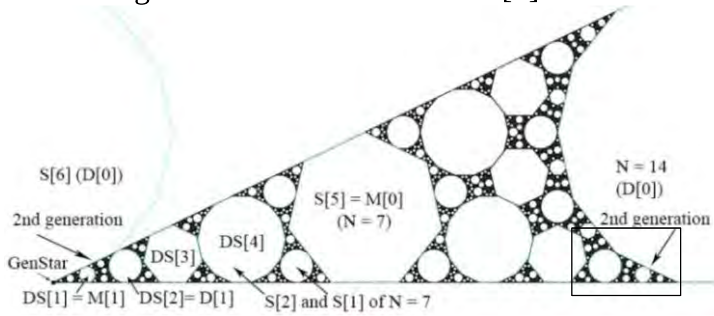
Therefore the similarity (box-counting) dimension of the three webs should be:
(i) N =5 &10: Log[6]/Log[1/GenScale[5]] ≈ 1.2411 where GenScale[5] = Tan[π/5]Tan[π/10]
(ii) N = 8: Log[9]/Log[1/GenScale[8]] ≈ 1.2465 where GenScale[8] = $\text{Tan}[\pi/8]^2$
(iii) N = 12: Log[27]/Log[1/GenScale[12]] ≈ 1.2513 where GenScale[12] = $\text{Tan}[\pi/12]^2$

For compact self-similar sets such as these, the similarity dimension will match the traditional Hausdorff fractal dimension. It is no surprise that these dimensions are increasing, but this applies only to the quadratic family. For the cubic family and beyond, the webs are probably multi-fractal – with a spectrum of dimensions. However it is likely that the maximal Hausdorff dimension will increase with the algebraic complexity of N, with limiting value of 2. See[LKV].

Below is a series of plots showing how the magenta web for N = 8 can be 'smoothly' mapped to the cyan web for N = 12, with N = 10 in between. These webs were generated by the digital-filter map of [H2] and the Appendix. This map creates webs which are clearly congruent to the outer-billiards map, but it allows for continuous variation of the angular parameter that defines N. That parameter is the same as the scaling field generator $\lambda = 2\cos\theta$ which here increases from $\sqrt{2}$ (2cos[2π/8]) to $\sqrt{3}$ (2cos[2π/12]) with N = 10 in between at 2cos[2π/10] = $(\sqrt{5}+1)/2$.



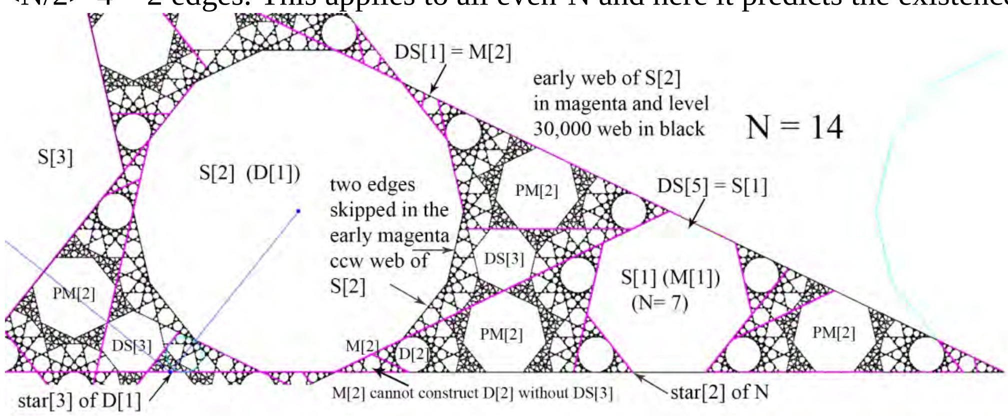
Example 5.3 (N = 7 & N = 14) This web plot would be unchanged if the origin was shifted to N = 7. The scaling fields $S_{14} = S_7$ are generated by GenScale[7] = Tan[π/7]·Tan[π/14] and this is also the scale of the 2nd generation at GenStar or star[1] of N.



This 2nd generation at GenStar or star[1] of N= 14 is clearly not self-similar to the 1st, but the even and odd generations appear to be self-similar and this dichotomy seems to be common for all N-gons with extended family structure at GenStar. Most cases are similar to N = 13 where the even and odd generations at GenStar are related in an imperfect fashion.

N = 7 and N = 9 are 'cubic' polygons so they have a second non-trivial primitive scale along with GenScale[N]. In both cases this competing scale is scale[2] = hS[1]/hS[2]. Even though the local web of S[2] will always generate S[1], their families have little in common and scale[2] is linearly independent of scale[3] (GenScale). In terms of web evolution, N = 7 evolves in a step-2 fashion so the S[1] and S[2] tiles of N = 7 are congruent to DS[2] and DS[4]. Therefore the edge geometry of N = 7 will involve tiles that are only distantly related.

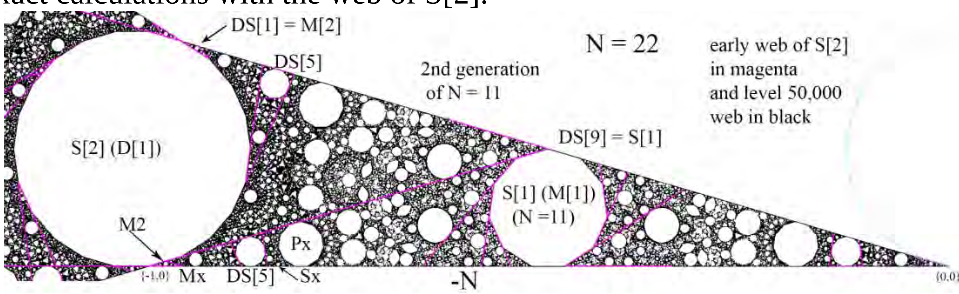
By contrast the 2nd generation dynamics on the edges of D or N = 14 are more manageable since S[2] and S[1] are now N-gons and a D-M pair. The local index of S[2] is N/2-2 = 5 so star[5] of S[2] is star[2] of N. In terms of edge skips this index is 4 so the retrograde ccw web of S[2] will skip <N/2>-4 = 2 edges. This applies to all even-N and here it predicts the existence of DS[1].



N = 14 is a member of the 8k+6 family so the Rule of 4 says that DS[1] will exist as an M[2] but apparently it cannot generate D[2] without help from DS[3] -as in the 8k+2 cases. Here the required DS[3] is on the left side of D[2] and the web shows that it does share an edge with the virtual D[2] in cyan.These even generations are dominated by weakly conforming PM tiles in place of the S[2]s of M[1]. This allows the edges of M[1] to develop 'normally' - unlike the edges of M. There is an dual orthogonal convergence at star[3] of D[1] with alternating PM and DS[3] tiles on the left and alternating virtual and real D[k] on the right. This makes the local geometry of star[3] a unique mix of scales. Click there to enlarge.

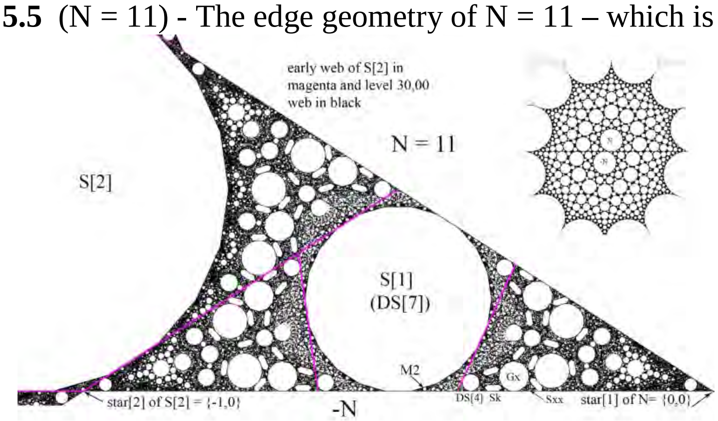
Example 5.4 (N = 11 & N = 22) The two-elephant case from Example 1.2 takes place in the 2nd generation of N = 11 so these tiles exist on the edges of D – which is a reflection of N = 22. This is an 8k+6 family like N = 14 above so there is a DS[1] serving as an M[2], but it is not capable of generating D[2] without the help of a DS[3] and it appears that the weakly conforming Mx tiles that appear in the 2nd generation are the remnants of DS[3]s which never formed.

The web plot below shows the predicted survival of M[2] and DS[5] – along with S[1] at DS[9]. This DS[5] played a role in the earlier construction of Sx and the other 'elephant' was Px which is only weakly conforming to D[1]. In [H6] we derive the parameters of Px along with Mx and Px. The difficult cases were Mx and Px and it was only possible to derive their parameters by doing exact calculations with the web of S[2].



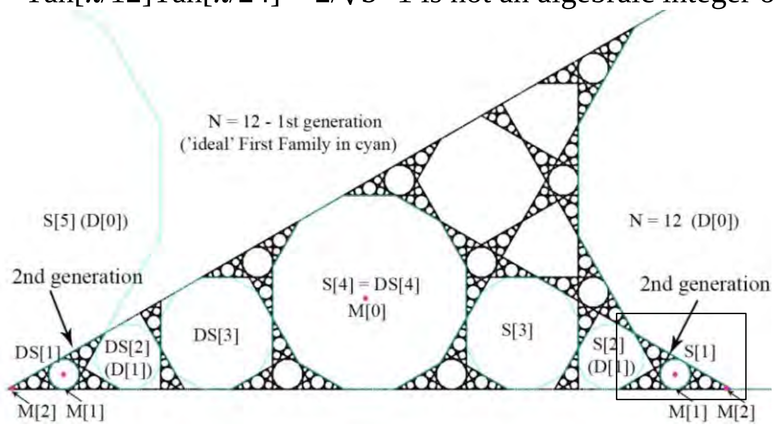
These calculations for Mx are repeated in the Appendix to show that the toral digital-filter map and the complex-valued dual-center map give the same parameters. We will often use the dual-center map in these web plots because it is easy to implement. This map necessitates a change of origin so that in the plot above, star[1] of N is at the origin and the edge length is 1. This allows a natural juxtaposition of N and –N as shown in the insert below for N = 11. The symmetry between these two representations is the key to the simplicity of this map. (The only difference between these webs and the normal τ-webs is the number of iterations needed. The dual-center webs are simpler but need more iterations because they are based on both N and –N.)

Example 5.5 (N = 11) - The edge geometry of N = 11 – which is in the 8k+3 family



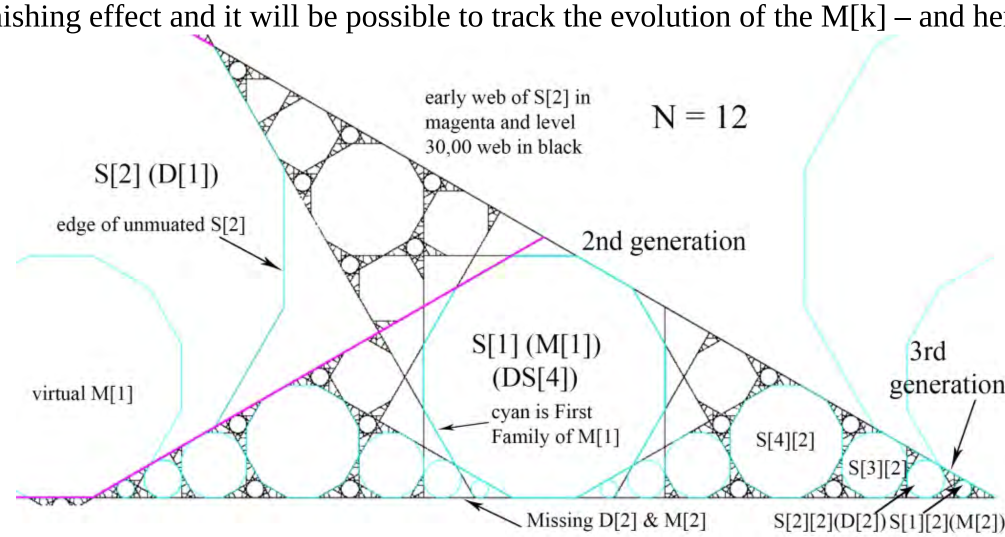
When N is odd, the 8k+1 and 8k+3 families have no canonical DS[1],DS[2] or DS[3] tiles. For N = 11 the only canonical tile of S[2] is S[1] at DS[7] so most of the structure is due to the evolution of S[1] – which is retrograde skip-3. This is consistent with DS[4]s which in turn generates M[2]s - but no matching D[2]s. There are also weakly conforming tiles that we call Gx's. In [H6] we find the parameters of Gx and the elongated hexagons which we call Sk.Using Gx and one of the Sk as 'elephants' it is possible to find the parameters of a small Sxx tile, but there is no clear sign of self-similarity or extended family structure.

Example 5.6 (N =12) For N twice-even the natural generation scaling is still through M[1] – which is identical to S[1] here. hM[1]/hN = Tan[π/12]^2 = $7-4\sqrt{3}$ = GenScale[12]. By contrast hS[2]/hN = Tan[π/12]Tan[π/24] = 2/$\sqrt{3}$ -1 is not an algebraic integer or a primitive scale.



The Mutation Conjecture says that S[k] mutations will occur here when gcd(N/2-k,N) > 2, so S[4] is not mutated but S[3] and S[2] are. Any mutation in S[2] is unusual and despite the extreme mutations of N = 24, S[2] is not mutated. Traditionally we have used S[2] to generate the local web on the edges of N-gons and we will do this here, but the geometry will be strange.

N = 12 is the first member of the 8k+4 family and the Edge Conjecture make no predictions other than S[1] at DS[4]. Since S[1] is the penultimate tile of D[1] it is also called M[1]. In cases like this it is fair to conclude that most of the web development will be driven by M[1] – but here M[1] has a local web which does not evolve. There are no other known cases of this, but it is clear that this web mutation only occurs for M[k] generated by mutated D[k]. This disruptive influence of the D[k] will continue for future generations but in each generation it will have diminishing effect and it will be possible to track the evolution of the M[k] – and hence the D[k].

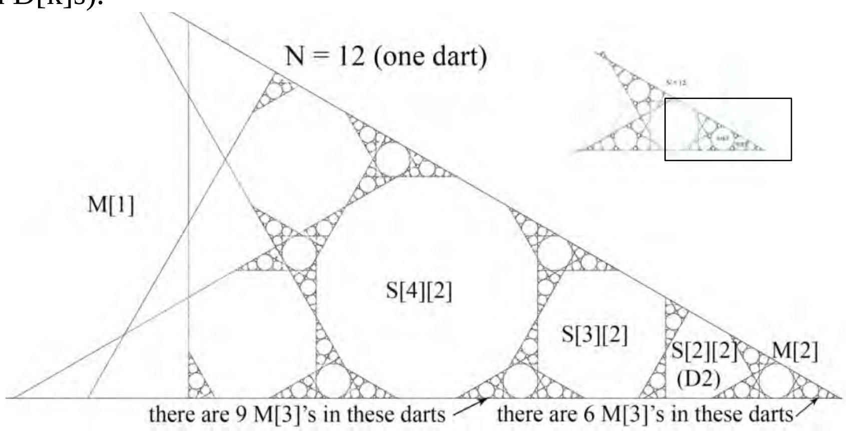


As indicated in Example 5.2, any τ-web based on a single primitive scale should be eventually self-similar and that appears to be the case here. The three 'darts' above are anchored by S[4]s and all the First Family tiles of M[1] scale by GenScale[12]. This determines the geometric scaling of the darts and assuming that these M[k] are dense, the only remaining issue is the 'temporal' scaling of the M[k]. The reasoning presented in Example 5.2 was that at each new generation the S[4] will scale by 3 and each S[4] is surrounded by 3 S[3]s so the S[3] scale by 9.

The non-trivial issue is to show that in the limit each S[3] accounts for 3 S[1]s – which are M[k]s. The M[2] count in the 2^{nd} generation above is 18 which is short of the predicted 27, but this is due to the mutated web of M[1] which excludes as many as 10 M[2]s.

Because of self-similarity, these mutations in the web of the M[k] will persist in future generations – but they only exist in the M[k] generated by D[k] and it is clear that 'most' M[k] are adjacent to the S[3] since they are step-3 tiles of the S[3]. This can also be observed in the First Generation above.

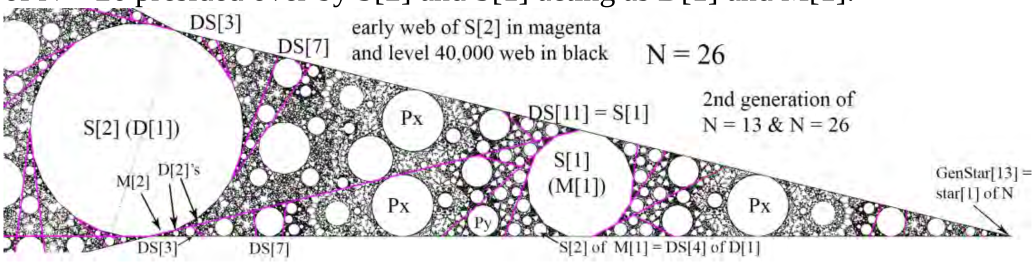
In the 2^{nd} generation dart enlarged below, the only M[2] that is generated by a D[2] is at star[1] of N. The remaining 5 M[2]s are generated as step-3 tiles of the S[3][2] so they have normal webs which will contain M[3]s at the step-1 positions. Therefore of the 23 'darts' shown here only 3 are affected by the mutations – and this ratio will decrease with each new generation to yield a limiting count of 3 M[k]s for each S[3][k-1] and a limiting temporal scaling of 27 for the M[k]s (and D[k]s).



Another way to verify the temporal scaling for self-similar webs is to simply count the growth of the tiles using the τ-periods. Here the τ-periods of the M[k] in the canonical invariant 'star-region' of Example 4.3 are 60, 942, 28292, 775356, 21055308,..with ratios of 15, 30, 27.41, 27.15. These are the combined periods of the M[k] at GenStar and their reflections about S[4]. Even though the local web has perfect reflective symmetry with respect to S[4], the dynamics are different and this combined count helps to minimize these differences. However the dynamics of any composite N-gon allows for 'decomposition' of expected orbits unto groups of orbits with smaller periods. This makes it difficult to match tile counts with periods, but for self-similar webs, the effect of these exceptions diminishes with each generation and in the limit the τ-ratios will match the geometric ratios.

It is easy to find these τ-periods because the M[k]s scale by GenScale[12]. Setting hN = 1 cM[k] = (1- GenScale[12]^k)·GenStar. For example the exact value of cM[12] is (1 - x^{12})·GenStar $=\{-4215120(-1694157+978122\sqrt{3}), -4215120(-6322680+3650401\sqrt{3})\}$.The matching M[12] at star[1] is a reflection of M[12] about cS[4]. In both cases the periods are so high that these points will generate very accurate webs tiled by microscopic M[12]s. In this GenStar or star[1] convergence the actual limit point does not have a well-defined orbit, but the astute reader may be able to do better and find a non-periodic interior point by tracing a sequence of M[k] tiles that do not converge to an edge except in the limit. Such points are easy to find for N = 5 and N = 8 but they are more challenging here.

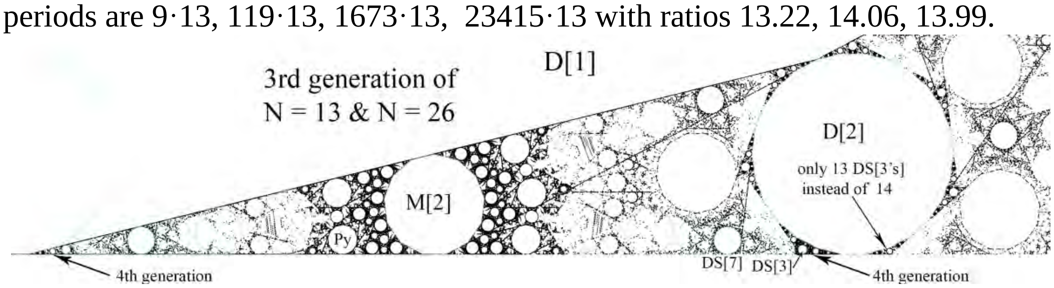
Example 5.7 (N = 13 & N = 26) N = 13 and N = 26 have algebraic order 6 along with N = 21,28,36 and 42. Since N = 13 and N = 26 have 6 primitive scales the web should be multi-fractal but the 4k+1 conjecture predicts self-similar sequences of M[k] and D[k] tiles converging to GenStar[13] at the foot of D - which is congruent to star[1] of N = 26. Therefore there may be some local self-similarity in the web. By convention we usually study this M[k[-D[k] sequence at the foot of S[2] – acting as D[1]. This sequence begins with D or N = 26 acting as D[0] and the matching M = M[0] which is S[11] (not shown here). Below is the 2nd generation on the edge of N = 26 presided over by S[2] and S[1] acting as D[1] and M[1].



N = 26 is an 8k+2 polygon and therefore DS[3] exists and can construct matching D[2]s and M[2]s in clusters separated by 2 blank edges as in N = 34 and N = 10 earlier. There will be k of these clusters on both sides of the line of symmetry and a shared cluster, which yields 4k+2 D[2]s = N/2+1. If this step-2 evolution continues for the new D[k] the temporal scaling of the D[k] should be 14, giving a local fractal dimension of Log[14]/Log[1/GenScale[13]] ≈ .7531.These local edge dimensions decrease with N with a minimum value of ½.

It is not difficult to find the parameters of weakly conforming tiles like Px. Py is not conforming to either D[1] or M[1] –but it is a ‘two-elephant’ case where Py shares star points with M[1] and a displaced S[2] of M[1]. See [H7] for a multi-generation derivation of Px and Py.

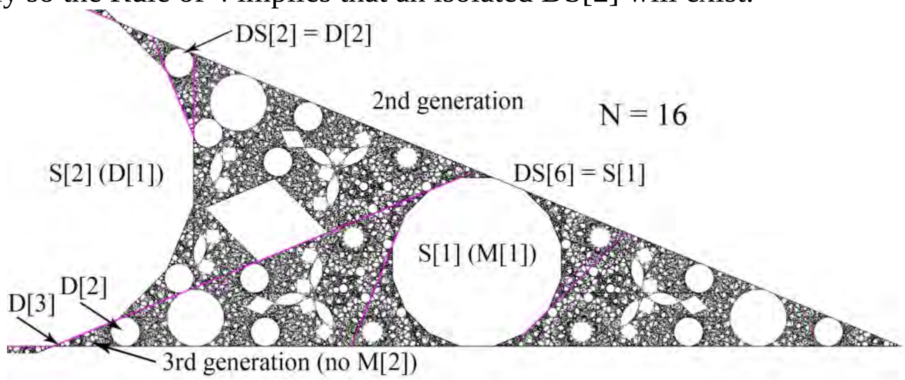
Below is the 3rd generation on the right side of D[1] but the left side of D[2]. This yields a ‘retrograde’ web relative to D[1] above, but it is still a classic 8k +2 case with canonical DS[3]s and DS[7]s, The reversed rotation entails a slight reduction of the resulting D[3]s because the shared DS[3] has only one D[3]. This appears to be generic for 4k +1 cases so the relative D[k] periods alternate low/high starting with D[1]. For N = 13, the D[1], D[2], D[3], D[4] global τ-periods are 9·13, 119·13, 1673·13, 23415·13 with ratios 13.22, 14.06, 13.99.



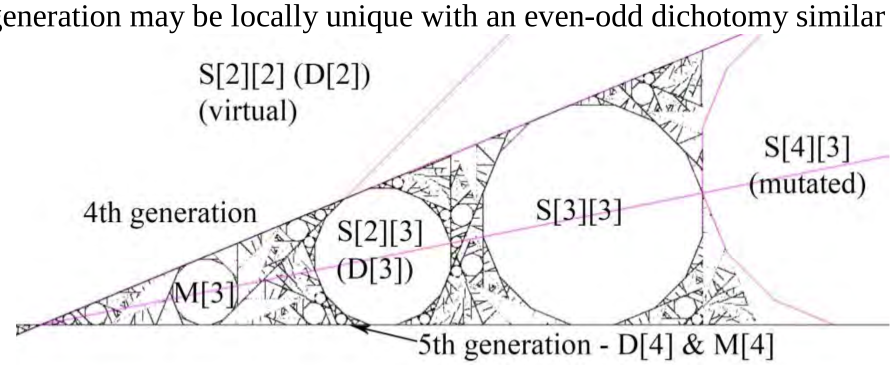
The first few generations of N=13 are described in [H7]. It is no surprise that the 4th generation is closer to the 2nd with both Px’s and Py’s. There does not appear to be perfect self-similarity between generations but the local geometry of M[1] seems to survive on all generations so there are Py[k] for all k but Px[k] only for k even. It is expected that these canonical Rule of 4 DS[k] develop consistent local webs, and this would explain why the M[k] have similar dynamics.

Example 5.8 (N = 16)

N = 16 has 'quartic' complexity along N = 15, 20, 24, and 30. Both N = 16 and N = 24 are in the 8k family so the Rule of 4 implies that an isolated DS[2] will exist.



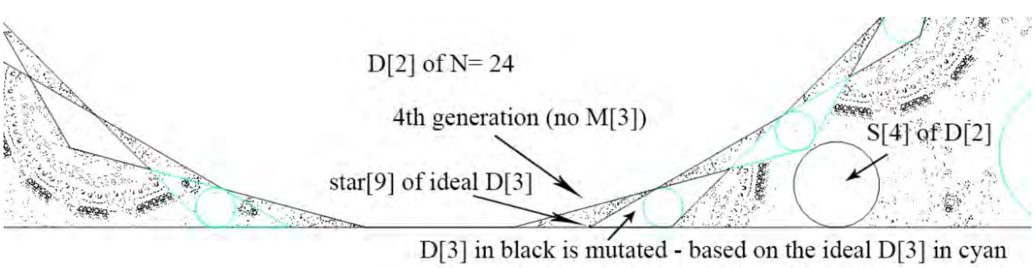
Even though M[2] is missing, the local web of D[2] contains a 'normal' M[3] and D[3] as well as matching S[3][3] and S[4][3] tiles as shown below. This makes the 3rd generation locally similar to the 1st generation. Since D[3] is clearly in the 8k dynamical family there is a well-defined 4th generation with M[4],D[4] and an S[4][4] – but no S[3][4] – so the 4th generation is only partially self-similar to the 2nd and 3rd. There is no doubt that this chain will continue and each generation may be locally unique with an even-odd dichotomy similar to N = 13.



The periods of the first 10 D[k] are: 8, 32, 456, 2464, 20872, 110368, 974664, 5165216, 45423368 and 240668192 which gives even and odd ratios of about 5.3 and 8.8.

N = 24 is also in the 8k family with an isolated D[2], but because of mutations, its local web is far from 'normal'. There are no M[3]s and only highly mutated D[3]s which are quadrilaterals based on ideal DS[3]s as shown below. It is possible for mutations to evolve in a self-similar manner as with N = 9 and N = 12, but there is no sign of that evolution here.

Example 5.9 (N = 24) - The 4th generation



Appendix. Exact Calculations using Symbolic Dynamics

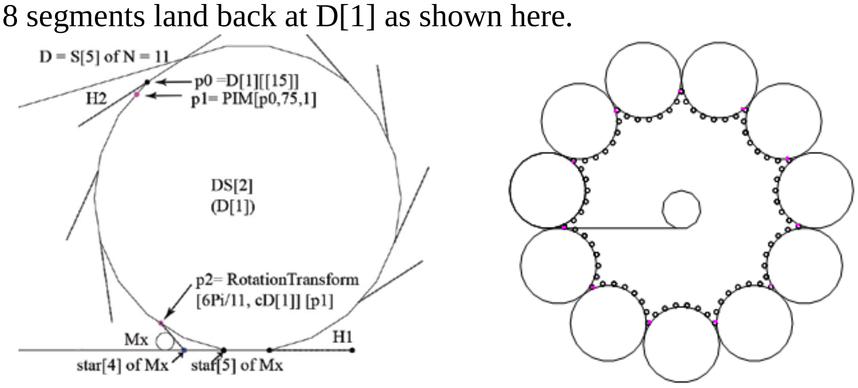
In [H2] we describe mappings from physics, astronomy, circuit theory and quantum mechanics which have singularity sets which may be conjugate to the outer-billiards map. Here we describe two such mappings and show how they can be used to perform exact calculations. The sample calculation will involve finding the parameters of the weakly conforming Mx tile of N = 11 – as described in Example 5.4. We will first solve this problem with τ using 'corner sequences'. These sequences are examples of 'symbolic dynamics' as first formulated by George Birkhoff and Stephen Smale.

The other two maps in question are the 'digital-filter' map of Chau & Lin [CL] and a refinement of a 'dual-center' map of Arek Goetz [Go]. Like the outer-billiards map, these are piecewise isometries based on rational rotations – also known as affine piecewise rotations. These three maps appear to have conjugate webs and this may be due to the fact that each can be reduced to a form of shear and rotation. Even though the webs are congruent, the dynamics appear to be very different but we shall show that they have a consistent form of symbolic dynamics.

Example A1 Use the evolution of the 'web' W to find the parameters of the Mx tile of N = 11 with (i) the outer-billiards map, (ii) the digital-filter map and (iii) a complex-valued Goetz map.

Part (i): The outer-billiards map. In Example 5.4, we noted that the Mx tile of N = 11 is a weakly conforming regular N-gon that occurs in the 2^{nd} generation for N = 11 on the edges of D. Therefore star[5] of Mx is star[1] of DS[1] as shown below. By the Two-Star Lemma, the parameters of Mx can be determined by finding another star point of Mx. In this example we will find the star[4] point of Mx by tracing the web evolution of the interval H1 - in the context of N with radius 1. (By reflection, this evolution can equally be studied on the edges of N = 22.)

The interval H1 lies on the horizontal base edge of N = 11 so there are 11 such intervals equivalent to H1 under rotation, and the local τ- web determined by H1 includes the iteration of each of these rotated copies. Under τ, these 11 regions map to each other and after 99 iterations of each interval, 8 segments land back at D[1] as shown here.



The interval H2 arises after just 13 iterations. Like all First Family vertices, p0 is in $\mathbb{Q}_{11}$ and we suspect that it maps to p1 (as a one-sided limit) – and hence determines the offset of p2. These vertices technically have no image under τ, so we will find p_1 using the 'surrogate' orbit of a point that is close to p0 and on the interval H2. (Typically intervals like H2 will get truncated under iteration, but by inspection it is clear that the inner portion of H2 survives well beyond the few hundred iterations needed here.)

Here are the calculations using Mathematica:

(i) Since the image under τ of any edge is a parallel edge, the slope of H2 is known, so set p0N to be a point on H2 within 8 decimal places of p0. **Orbit = NestList[τ, p0N, 200]** (an approximate orbit but initially reliable and any errors are easy to detect).

(ii) Since $\tau(p) = 2c_j - p$ for some vertex c_j of N = 11, $\tau^k(p) = (-1)^k p + 2Q$ where Q is an alternating sum of vertices. Every τ-orbit determines a sequence $\{c_k\}$ of vertices and the matching indices are sufficient to find Q and determine the orbit. The study of these partition sequences is called 'symbolic dynamics' so we will call them S-sequences. The Mathematica module IND will use τ to find the S sequences to any depth (once again with possible error).

S[p0N,150] = **IND[p0N,150]** = {11,5,10,4,9,3,8,1,6,11,5,10,4,9,2,7,1,6,11,5,…}

Note that these indices initially advance by {5,5,5,5,5,5,4} (mod 11) because D is S[5] with step sequence {5} and D[1] has (periodic) step sequence {5,5,5,5,5,5,4}. Here this sequence will eventually break down. In general no web point can have a periodic orbit because these points have no inverse. We will use these indices in pairs, using the 'return' map $\tau^2(p) = p + 2(c_k - c_j)$.

(iii) P1 = **PIM[p0N,75,1]** will take IND and these 150 indices in pairs and reconstruct the orbit – while P3 = PIM[p0N,75,3] will construct a step-3 version of this orbit – which is called a 'projection' or algebraic graph as defined in [S2]. To get an exact orbit, simply use p0 instead of p0N.

P1 = PIM[p0,75,1] ; p1= P1[[75] = τ^{150}(p0) ; p1[[1]] =

$$-6Cos[\frac{3\pi}{22}] - Cos[\frac{\pi}{11}]Cot[\frac{\pi}{22}] + 4Sin[\frac{\pi}{11}] + 8Sin[\frac{2\pi}{11}] + Cot[\frac{\pi}{22}]Sin[\frac{\pi}{11}]Tan[\frac{\pi}{11}] - Sec[\frac{\pi}{22}]Sin[\frac{\pi}{11}]Sin[\frac{5\pi}{22}]Tan[\frac{\pi}{11}]$$

(iv) Rotate by 6π/11 about the center of D[1]: p2= **RotationTransform[6·Pi/11, cD[1]][p1]**

(v) As indicated earlier, the slope of the web interval determined by p2 must match an edge of N. Here it has same slope as the (right-side) star[4] edge of N – which we call slope4.
x1 = p2[[1]], y1 = p2[[2]]; b = y1-slope4·x1 so star[4][[1]] = (1- b)/slope4

(vi) By the Two-Star Lemma hMx = d/(Tan[5Pi/11]-Tan[4Pi/11]) where d is the horizontal displacement of star[4] and star[5].

(vii) Of course the displacement d depends of hN but that dependence vanishes when hMx is divided by hN – which here is Cos[π/11].

AlgebraicNumberPolynomial[ToNumberField[hMx/hN,GenScale[11], x] =

= $1 - 23x - \frac{27x^2}{2} + \frac{x^4}{2}$ where x = GenScale[11] = Tan[π/7]·Tan[π/14].

Any other hN and matching hMx must yield this same ratio so this is a fundamental polynomial for Mx. Any 'canonical' tile with scaling in S_N will have such a polynomial. See Example A2.

Part (ii) – The Digital Filter Map. The Df map is only compatible with the outer-billiards map when N is even, so it is necessary to work inside N = 22 (with hN = 1). This is not a burden and actually simplifies the calculations. Except for scale, the First Family is unchanged from part (i) - but S[9] is now playing the part of N = 11, so we will show that hMx/hS[9] satisfies the polynomial above.

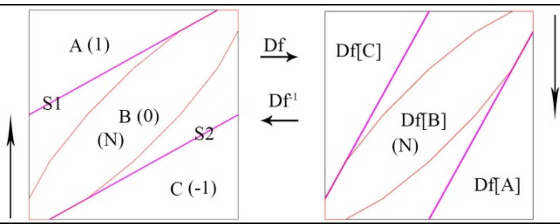
The Digital Filter map Df: $[-1,1)^2 \to [-1,1)^2$ is defined as Df[{x,y}]:={y, *f* (-x + *a*y)} where *f*(v) = Mod[v+1,2]-1 models a 2's complement sawtooth register. In matrix form (where *f*(y) ≡ y):

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = f\left(\begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix}\begin{bmatrix} x_k \\ y_k \end{bmatrix}\right) = \begin{bmatrix} y_k \\ f(-x_k + a y_k) \end{bmatrix}$$

Setting $a = 2\cos\theta$ the matching elliptical rotation

$\begin{bmatrix} 0 & 1 \\ -1 & 2\cos\theta \end{bmatrix}$ is conjugate to a true rotation $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ but this conversion is optional.

To generate the web of N = 22, set θ = 2π/22 and here we are interested in generating symbolic orbits. The S sequences will be much simpler than τ, because the function *f* distinguishes just 3 regions - so Df is a piecewise isometry with three primary regions (atoms) which can be labeled 1 (overflow), 0 (in bounds), or -1 (underflow).The equations for these atoms are given here.

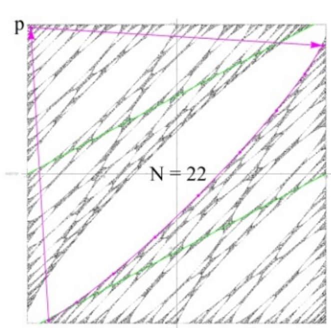


$$S[\{x,y\}] = \begin{cases} 1 & \text{if } -x + ay \geq 1 \\ 0 & \text{if } -1 \leq -x + ay < 1 \\ -1 & \text{if } -x + ay < -1 \end{cases}$$

Under Df all points experience a rotation by θ and *f* determines the corresponding 'vertical' shear of -2, 0 and 2 respectively for A, B and C. The central B region is free of translation since it is 'in-bounds', so points will rotate by θ, and under iteration they will construct copies of N – one of which is shown here. The seperatices S1 and S2 define the maximal extent of this linear (elliptical) rotation, so they define the bounds of the three regions. Therefore if the S-sequence of a point p is known the Df map is simply DfS[{x,y},k_]:={y,-x+*a*y-2S[[k]]}

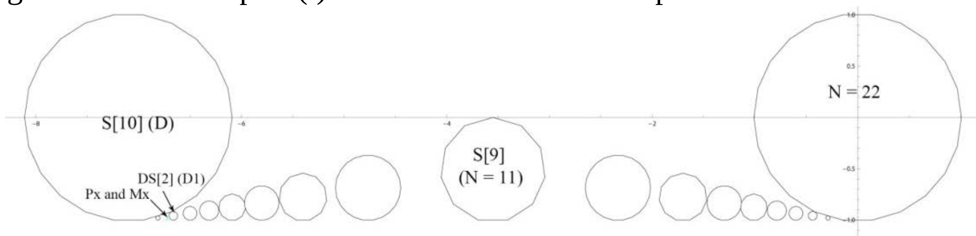
Example: The first 10 points in the Df orbit of cDS[1] for N = 22. Set p = TrToDf[cDS[1]] where TrToDf is a change of coordinates between traditional Euclidean space and Df toral space.

Like all points adjacent to D, p ={x,y} will be in an 'overflow' so Df[p] = {y,-x+*a*y-2} as shown here. The point {y,-x+*a*y} is 'out of bounds' at top right and the {0,-2} shear is the *f* correction. Since Df[p] is 'in-bounds', it allows for 8 consecutive central rotations, but the last rotation will yield an 'underflow' which will generate a compensating displacement back to an overflow position to repeat the cycle. Therefore the S sequence will be {1,0,0,0,0,0,0,0,0, -1,1} and this periodic sequence determines the orbit of p.

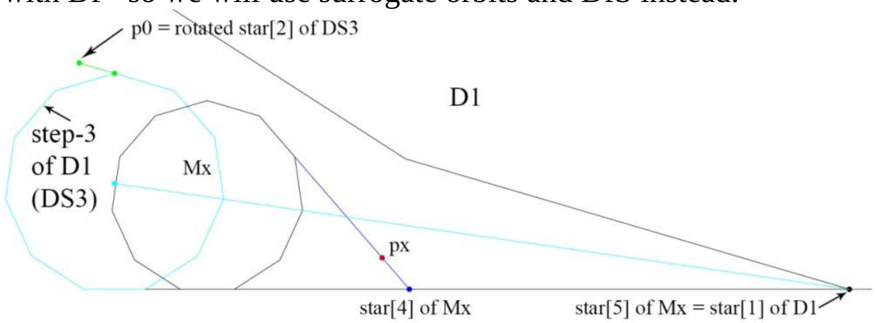


To get the level-k S sequence of any point p, first generate Orbit = NestList[Df,p,k] and then apply the function S to the elements of this orbit: Sequence = S/@Orbit.

As a guide we will use part (i) above to construct the expected Mx tile inside N = 22



The closest connection between Mx and the First Family of D1 appears to be the (virtual) step-3 tile of D1 – which we call DS[3][2] or simply DS3. We will use DS3 to define an interval that maps to Mx. The green interval shown below is a rectified portion of an edge in W, and under Df this interval will map to the blue interval – when rectified. Therefore p0 maps to px (using Df) and p0 is exact. It only takes 136 iterations to accomplish this – but an exact calculation will be awkward with Df - so we will use surrogate orbits and DfS instead.



Here are the calculations :

(i) Use p1 = TrToDf[p0] to generate an approximate Df orbit of length 140 using w = 2Cos[2Pi/22] to 30 decimal places. Orbit=NestList[Df, p1,140]; S=S/@Orbit ={1,0,0,0,0,0,0,0,0,-1,1,0,0,0,…} (As indicated earlier the points at the foot of D are typically in overflow positions so sequences such as this are common. Underflow must map to overflow because -1 and 1 are only one bit apart in 2's complement.)

(ii) Generate an exact value of Df^{137}[p1] using DfS and exact w = 2Cos[2Pi/22]. Set q=Range[140]; q[[1]]= p1; **For[s=1, s<=140, s++,q[[s+1]]=Simplify[Re[DfS[q[[s]],s]]]];**

pxx (px unrectified) = q[[137]] ≈ {-6.679854552975, -0.9991754019926052}

px = Simplify[Re[DfToTr[pxx]]]; px[[1]] =

$$(-422+419(-1)^{1/11}-414(-1)^{2/11}+419(-1)^{3/11}-422(-1)^{4/11}+421(-1)^{5/11}-420(-1)^{6/11}+424(-1)^{7/11}-424(-1)^{8/11}$$
$$+420(-1)^{9/11}-421(-1)^{10/11})/((1+(-1)^{2/11})(-1+(-1)^{1/11}+(-1)^{6/11})^2)$$

(iii) Set x1 = px[[1]]; y1 = px[[2]]; b = y1-slope4·x1 (where slope4 is the slope of right-side star[4] of N – as in part (i)). Therefore star[4] [[1]] = (1-b)/slope4 =

$$\frac{i(-13-(-1)^{1/11}+20(-1)^{2/11}+14(-1)^{4/11}-14(-1)^{5/11}-20(-1)^{7/11}+(-1)^{8/11}+13(-1)^{9/11})}{(1+(-1)^{1/11})(1+(-1)^{2/11})^4}-2Cot[\frac{\pi}{11}]$$

(iv) As in part (i), hMx = d/(Tan[5Pi/11]-Tan[4Pi/11]) where d is the horizontal displacement of star[4] and star[5].

(v) This displacement d is relative to hN so the ratio hMx/hN is the scaling field S_{11}, but N is a 22-gon, so it makes more sense to use hMx/hS[9] where hS[9] = Tan[π/22]/Tan[π/11].

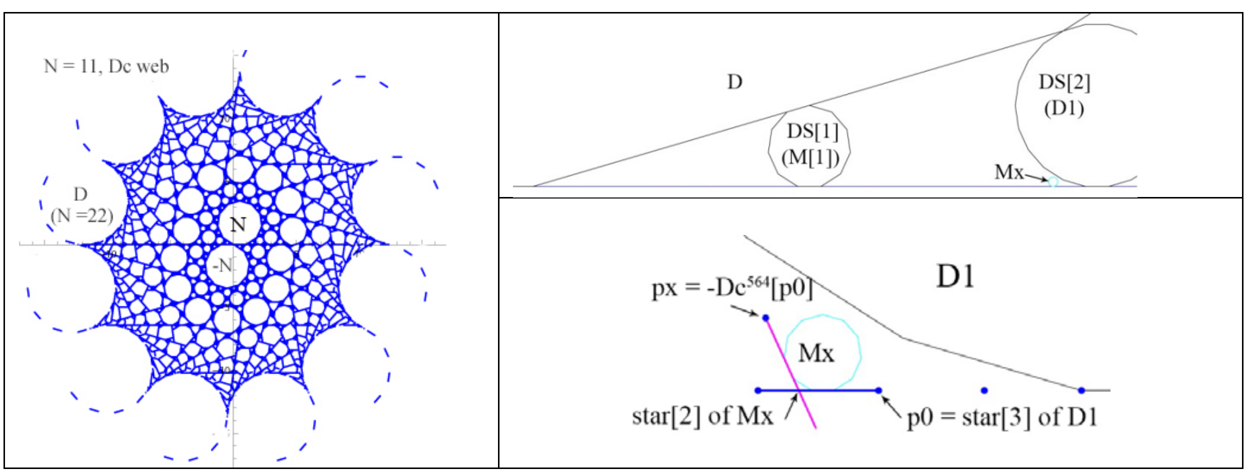
AlgebraicNumberPolynomial[ToNumberField[hMx/hS[9], GenScale[11]],x] gives

$1-23x-\frac{27x^2}{2}+\frac{x^4}{2}$ as in part (i) above.

Part (iii) A complex-valued Goetz Map

Perhaps the simplest map that reproduces the outer-billiards web for a regular N-gon is a y-axis version of a 'dual center' map of Arek Goetz. As described in [H2] this mapping has the form Dc[z] = Exp[-Iw]·(z - Sign[Im[z]) where w = 2π/N and for a scalar v , Sign[v] = 1, 0 or -1 iff v is positive, 0 or negative. Therefore Dc is a pure (clockwise) rotation for points on the x-axis but in general it is a 'shear and rotate' where the shear has magnitude -1 above the x-axis and +1 below (in a manner similar to W with sN = 1). So if Dc is applied to the interval [-1,1] it will form a perfect N-gon above the x-axis and its negative below. If this initial interval is expanded as shown below for N = 11 (with w = 2π/11), then the edges of –N will intersect the edges of N to simulate the interaction of the τ-domains. This juxtaposition actually occurs with W but not at the origin. Click here to see it for N = 11 and N = 22.

The web on the left below was generated by iterating the x-axis interval [-12,0], 200 times under Dc with w = 2π/11 (to 30 decimal places). In the limit, the region above (or below) the x-axis will be a perfect reproduction of the web for N = 11 – with a side of 1 and star[1] at the origin. It is an easy matter to scale the First Family (and Mx) to use as guides to track the web development at D. On the right we show a portion of the ideal second generation in black - and the hypothetical Mx in cyan. Unlike the Df map, the webs tend to evolve is a predictable fashion from intervals on the x-axis - and this makes it easy to find intervals that will map to Mx. The chosen interval is shown in blue in the enlargement on the lower right, and the magenta interval is the negative of the image of this blue interval under Dc^{564} .



Since the slope of this magenta line is known, all that is needed to find star[2] of Mx is px. Here are the calculations.

(i) The initial point is p0=StarD1[[3]] =

$$\{\frac{5}{2}-Cot[\frac{\pi}{22}]Cot[\frac{\pi}{11}]+Tan[\frac{\pi}{22}]Tan[\frac{\pi}{11}]+\frac{1}{2}Cot[\frac{\pi}{22}]Cot[\frac{\pi}{11}](1-Tan[\frac{\pi}{22}]Tan[\frac{\pi}{11}](2+Tan[\frac{\pi}{22}]Tan[\frac{\pi}{11}]))-(\frac{1}{2}Cot[\frac{\pi}{11}]+\frac{1}{2}Cot[\frac{\pi}{11}](-1+Tan[\frac{\pi}{22}]Tan[\frac{\pi}{11}](2+Tan[\frac{\pi}{22}]Tan[\frac{\pi}{11}])))Tan[\frac{3\pi}{22}],0\}$$

(ii) Using p0N = N[p0,30] (30 decimal place approximation to p0), find the first 600 points in the (complex-valued) orbit: Orbit = NestList[Dc,p0N,600]

(iii **S[z_]:=Sign[Im[z]];** Sx=S/@Orbit = {0,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,1,1,1,1,1,…}

(iv) Define a 'literal' version of Dc based on Sx , namely **DcS[z_,k_]:=Exp[-I·w](z-Sx[[k]])** (with an exact w) to obtain the exact orbit of p0 using the Sx sequence from the surrogate orbit. Store the orbit in the sequence q:

q=Range[600];q[[1]]=p2[[1]]; For[s=1,s<=600, s++, q[[s+1]]=Simplify[DcS[q[[s]],s]]];

Note: By modifying F to allow approximate calculations of the Sign function, it is may be feasible to do calculations like this directly with NestList. One possible modified function is F[z] = Simplify[Exp[-I*w](z – I*IntegerPart[Sign[N[Re[z]]]])]. This tricks Mathematica into regarding the Sign output as exact. Normally Mathematica attempts to evaluate Sign in an exact fashion and these expressions get so complex that typically it fails after a few hundred iterations.

(v) q[[564]] = $\frac{6-8(-1)^{1/11}+5(-1)^{2/11}-2(-1)^{3/11}-(-1)^{4/11}+3(-1)^{5/11}-(-1)^{6/11}+5(-1)^{7/11}-3(-1)^{8/11}+(-1)^{9/11}+4(-1)^{10/11}}{-1+2(-1)^{1/11}-3(-1)^{2/11}+3(-1)^{3/11}-2(-1)^{4/11}+(-1)^{5/11}}$

The desired point is px = - q[[564]] ≈ -10.415046959467414 + 0.01643324914196498·I

(vi) x1 = Re[px]; y1 = Im[px]; The slope of star[2] of Mx is –slope2 as defined using star[2] of N, so b = y1+ slope2·x1 and star[2] [[1]] = b/slope2 ≈ -10.407542146047456045

(vii) Using the Two-Star Lemma with opposite sides hMx = d/(Tan[2Pi/11]+Tan[5Pi/11]) where d = star[5][[1]]-star[2][[1]].

(viii) Since N has side 1, hN = $\frac{1}{2}Cot[\frac{\pi}{11}]$. Use this to convert hMx to a scale in S_{11}.

AlgebraicNumberPolynomial[ToNumberField[hMx/hN],GenScale[11]],x]

= $1-23x-\frac{27x^2}{2}+\frac{x^4}{2}$ as in parts (i) and (ii)

Since the star[3]-star[4] interval of D1 generates Mx, it can be regarded as a 'mutated' DS3[2]. Likewise Px is generated by the star[6]- star[7] interval so it may be a modified DS6[2].

The historical connection between these maps is illustrated in Example A3 below – which shows that Df is equivalent to a sawtooth (tuned) version of the classical Standard Map while Dc is a sawtooth (tuned) version of a kicked harmonic oscillator. The connection is that the Standard Map is equivalent to a kicked 'free' rotor in zero gravity while a harmonic oscillator is assumed to be affected by gravity – and hence has a natural frequency of oscillation.

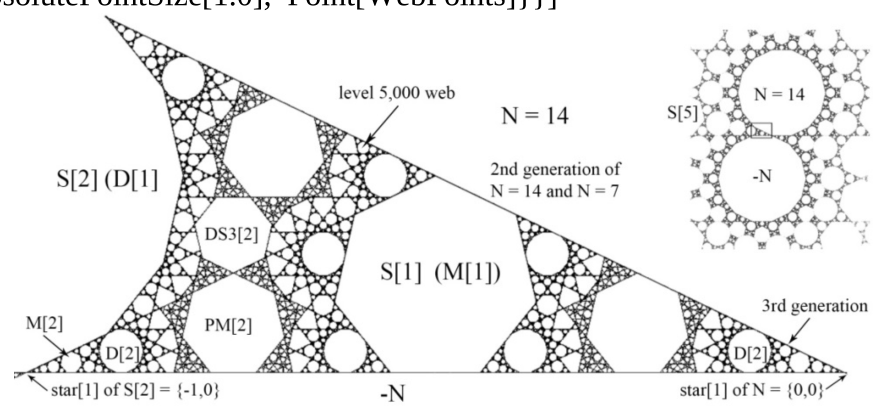
The classical Twist Map of J. Moser is also a 'free rotor' but it allows for the possibility of periodic kicks so the Standard Map illustrates the case of a Twist Map with these (ideal) periodic 'gravitational perturbations' turned on - and the issue is whether there will be a non-zero measure of initial conditions which yield 'integrable' solutions to the matching Hamiltonian.

It is still not clear what role gravity plays in quantum mechanics. In the late 50's P.B. Harper and others used a kicked harmonic oscillator with a natural frequency of oscillation to model both classical and quantum diffusion - based on a stationary Schrodinger equation. This is illustrated in Example A3 for $\omega = 2\pi/4$ (N = 4) which shows initial resonances similar to the Standard Map – and then diffusive breakdown and weak mixing as K increases.

The Df and Dc maps are very efficient ways to generate the local web for a regular N-gon but Dc has a number of advantages and we will be using it to publish a detailed '4K' catalog of edge geometry in [H9]. For this purpose the juxtaposition of N and –N is ideal and the natural +/- symmetry of W is augmented by reflective symmetry to yield very efficient maps.

Below is an example from N = 14 where we iterate 1,000 points in the interval H = {-2,-1} at a depth of 5,000. (The interval {-1,1} will generate N and –N in a period N orbit.) Here we crop these 5 million web points and their negatives and reflections to the desired region. (Less than 1 minute to generate and 1 minute to crop on a modest computer.)

Example A2 (The edge geometry of N = 14) Dc[z_]:=Exp[-I*w]*(z-Sign[Im[z]]);
w =N[2*Pi/14, 35];(35 decimal places); H=Table[x,{x, -2, -1, .001}]; Web =
Table[NestList[Dc, H[[k]], 5000],{k, 1, Length[H]}]; RealWeb = {Re[#], Im[#]}&/@Web;
WebPoints = Crop[Union[RealWeb, -RealWeb, Reflection[RealWeb]]]; (about 450,000 points)
Graphics[{AbsolutePointSize[1.0], Point[WebPoints]}}]

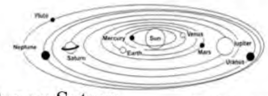


Example 5.3 discussed the N = 14 and N = 7 families, and here we examine the scaling. The scaling fields S_7 and S_{14} are generated by x = GenScale[7] = Tan[π/7]·Tan[π/14]. Inside N = 14, S[5] is the surrogate N = 7, so by convention all heptagons are scaled relative to S[5]. M[1] (a.k.a.M1) is a 2^{nd} generation N = 7 , but the edge dynamics are much simpler than N = 7 because S[2] is missing. M[2] is the 'matriarch' of the 3^{rd} generation and has the same edge geometry as N = 7. The PM tiles are only weakly conforming to D[1]. See [H8].

hM1/hS[5]	hD1/hN	hDS3[2]/hS[5]	hPM[2]/hS[5]	hS[5]/hN	hM2/hS[5]	hD2/hN
x	x	$\frac{1}{2}-4x-\frac{3x^2}{2}$	$-\frac{3}{8}+\frac{17x}{4}+\frac{9x^2}{8}$	$\frac{3}{7}+\frac{3x}{7}-\frac{x^2}{7}$	x^2	x^2

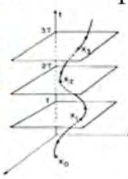
Example A3 - Some classical maps relevant to this paper

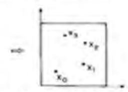
Twist Maps (zero gravity)



Jupiter vs Saturn

Poincare Section



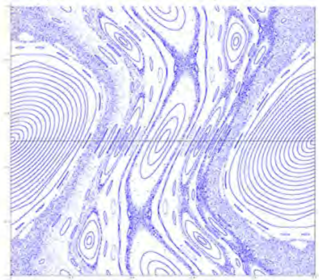


Twist Map ([Mo1] 1962)

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k + x_k \end{bmatrix}$$

(Here y is θ and x is momentum (constant))

Standard Map (Chirikov 1969) (K = .971635406)

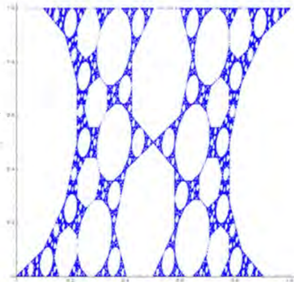


This is a 'Mod -1' periodically perturbed Twist Map showing resonant 'islands' of stability on all scales

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k + (K\sin 2\pi y_k)/2\pi \\ y_k + x_k + (K\sin 2\pi y_k)/2\pi \end{bmatrix}$$

(gravity turned on)

Sawtooth Standard Map ([A] 2001) (k = 2Cos(2Pi/14)-2)



The S[k] of N = 14 are the primary resonances of this Standard Map tuned to 'Ch14'

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k + k\mathrm{Saw}(y_k) \\ y_k + x_k + k\mathrm{Saw}(y_k) \end{bmatrix}$$

(Saw(x) = x - 1/2 on [0,1])

(The outer-billiards web W can be regarded as a sawtooth (tuned) version of the Standard Map)

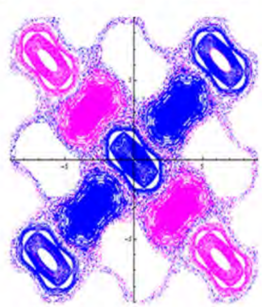
These are equivalent under a change of variables

Harmonic Oscillator (non-zero gravity)

natural frequency of oscillation ω

As in the Standard Map there may be periodic 'kicks' (and possible friction)

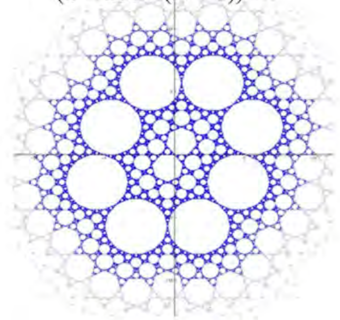
Kicked Harmonic Oscillator (KHO) Harper Map (ω = 2Pi/4) ([H] 1955)



$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x_k + K\sin y_k \\ y_k \end{bmatrix}$$

(sometimes unequal kicks are applied to both x and y)

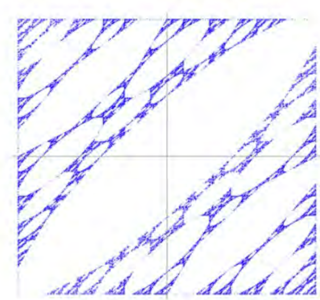
Dissipative KHO (DKHO) ([H2] 2012) (ω = 2Pi/7 (N = 7))



$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix} \begin{bmatrix} x_k - \mathrm{sgn}\, y_k \\ y_k \end{bmatrix}$$

Replacing sine with sgn (sign) yields a sawtooth (tunable) map that we call the dual-center map. In complex form: F[z] = Exp[-Iω]*(z- Sign[Im[z]])

Digital Filter Map ([CL] 1988) (a = 2Cos(ω) - here ω =2Pi/14

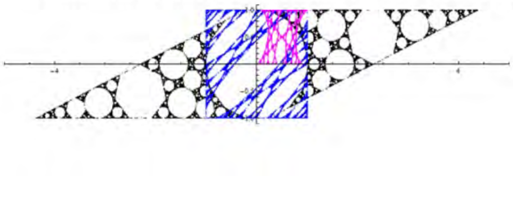


$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = f\left(\begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} \right) = \begin{bmatrix} y_k \\ f(-x_k + a y_k) \end{bmatrix}$$

Since a =2Cos(ω) this is conjugate to $\begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}$

The (sawtooth) register overflow function is $f(x)$ = Mod[x+1,2] -1 and $f(y_k) = y_k$ because y is always in range

Overlay of Sawtooth Standard Map (magenta), Digital Filter Map (blue) and rectified N = 14 web in black



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Links

(i) The author's web site at DynamicsOfPolygons.org is devoted to the outer billiards map and related maps from the perspective of a non-professional.

(ii) A Mathematica notebook called FirstFamily.nb will generate the First Family and related star polygons for any regular polygon. It is also a full-fledged outer billiards notebook which works for all regular polygons. This notebook includes the Digital Filter map and the Dual Center map. The default height is 1 to make it compatible with the Digital Filter map. This full-fledged notebook is not necessary to implement the Digital Filter or Dual Center maps – but it is useful to have a copy of the matching First Family to be used as reference.

(iii) Outer Billiards notebooks for all convex polygons (radius 1 convention for regular cases). There are four cases: Nodd, NTwiceOdd, NTwiceEven and Nonregular.

(iv) For someone willing to download the free Mathematica CDF reader there are many 'manipulates' that are available at the Wolfram Demonstrations site - including an outer billiards manipulate of the author and two other manipulates based on the author's results in [H2]. At the DynamicsOfPolygons site there are cdf manipulates at Manipulates – which can be downloaded. (The on-line versions have been phased out by most web browsers for security reasons.)