GeneticsOfPolygons.org

Affine transformations

Affine transformations and affinely equivalent polygons

An affine transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation together with a possible translation so it has the form: $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} = \mathrm{A}x + t$ where we make no distinction between {e,f} and $\begin{pmatrix} e \\ f \end{pmatrix}$

When A is invertible, these transformations form a group called the affine group. This is equivalent to the requirement that Det[A] $\neq$ 0. When Det[A] > 0 the transformation also preserves orientation. We will assume that all affine transformations have Det[A] > 0

In Mathematica:

$\mathrm{T}=\mathrm{AffineTransform}[\{\{\{a_{1,1},a_{1,2}\},\{a_{2,1},a_{2,2}\}\},\{b_1,b_2\}\}]$ will yield T[{x,y}]= $\{b_1 + xa_{1,1} + ya_{1,2}, b_2 + xa_{2,1} + ya_{2,2}\}$

The most common affine transformations are rotations,scaling (including reflections) and shears. All triangles are affinely equivalent and this is true for parallelograms as well, but in general quadrilaterals are not affinely equivalent, so a Penrose kite is not affinely equivalent to a trapezoid.

For a given point p, the tangent map, Tau is an affine transformation consisting of a reflection followed by a translation. As such it commutes with any orientation preserving affine transformation. (If the transformation reverses orientation, Tau becomes Tau^{-1}.)

Therefore all affine transformations of a given polygon have conjugate dynamics and we will call them 'affinely equivalent'.The affinely equivalent regular polygons will be called 'affine regular polygons'.

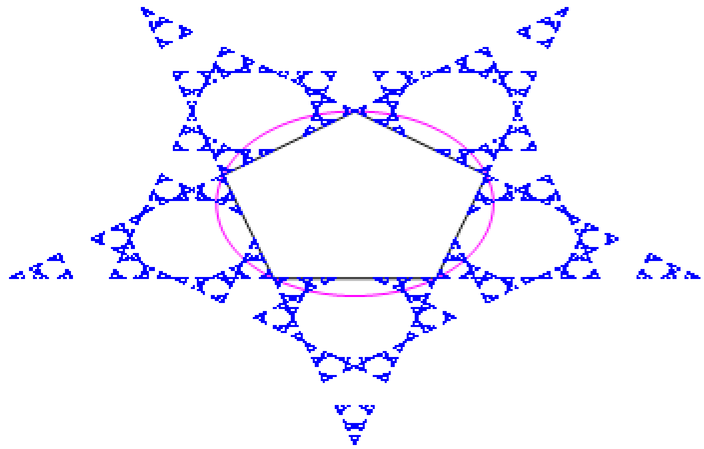
Example 1: An elliptic regular pentagon: **T = AffineTransform[{{3,0},{0,2}}]**

Mom = Table[{Sin[2 Pi n/5],Cos[2 Pi n/5]},{n, 0, 4}] (*regular pentagon with vertex 1 at {0,1}*)

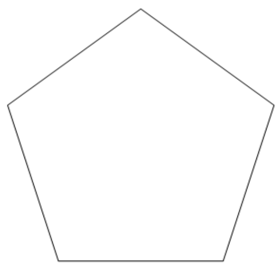
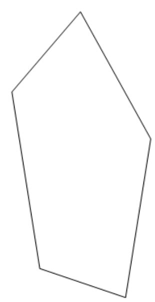
T[Mom] (*elliptic regular polygon with vertex 1 at {0,2}*)

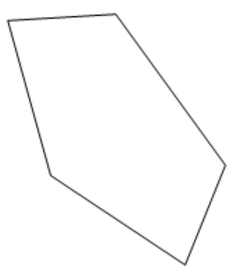
p = {T[Mom][[5]][[1]], T[Mom][[4]][[2]]} (* p =T[q] where q is non-periodic in regular polygon, N = 5 *)

Graphics[{poly[T[Mom]], Magenta, Line[ellipse], Blue, AbsolutePointSize[1.0], Point[NestList[Tau,p,30000]}]



Example 2: **T = AffineTransform[{{.184, -.6},{1.16, .6}}];**

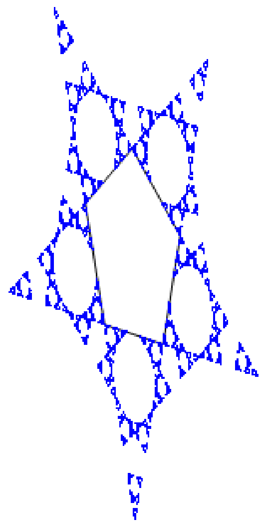
Mom  ⟶ **T[Mom]**  ⟶ **T[T[Mom]]**



Tau◦T = T◦Tau as shown below

p1≈ {-0.951056516295153572,-0.809016994374947424 10} (*non-periodic point for regular polygon N = 5 *)
T[p1] ≈ {0.31041579762666016, -1.5886357555273465} (*non periodic point for T[Mom] *)

Graphics[{AbsolutePointSize[1.0], Blue, Point[NestList[Tau, T[p1], 30000]]}]



An affine transformation is a special case of a *linear fractional transformation* that maps z to (az+b)/(cz+d)

Example: **a = {{1, 2}, {3, 4}};b = {5, 6}; c = {7, 8}; d = 17;**

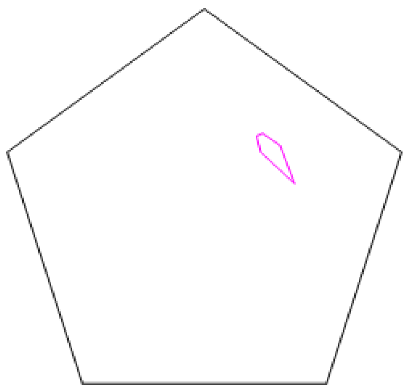
T= LinearFractionalTransform[{a, b, c, d}]; T[{x,y}]= $\left\{\frac{5+x+2y}{17+7x+8y},\frac{6+3x+4y}{17+7x+8y}\right\}$

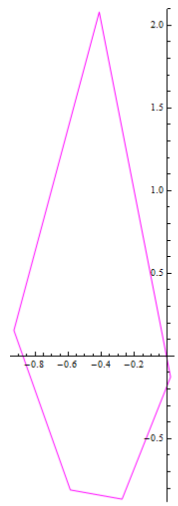
In the complex plane T[z] = (az+b)/(cz+d) is called a *Mobius transformation* after August Mobius. (By convention it is assumed that , ad-bc $\neq$ 0 because otherwise T[z] is constant.) T can be extended to the entire Riemann sphere in the obvious way and this becomes a holomorphic transformation from the Riemann sphere to itself.

Example 3: The (non-affine) linear fractonal transformation from above applied to the regular pentagon

T[{x,y}]= $\left\{\frac{5+x+2y}{17+7x+8y},\frac{6+3x+4y}{17+7x+8y}\right\}$

Graphics[{poly[Mom], Magenta, poly[T[Mom]]}] (* the scaled and rotated version is on the right*)





Below is a portion of the web: